

Stochastic Differential Games with Reflection and Related Obstacle Problems for Isaacs Equations

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Abstract In this paper we first investigate zero-sum two-player stochastic differential games with reflection with the help of theory of Reflected Backward Stochastic Differential Equations (RBSDEs). We will establish the dynamic programming principle for the upper and the lower value functions of this kind of stochastic differential games with reflection in a straight-forward way. Then the upper and the lower value functions are proved to be the unique viscosity solutions of the associated upper and the lower Hamilton-Jacobi-Bellman-Isaacs equations with obstacles, respectively. The method differs heavily from those used for control problems with reflection, it has its own techniques and its own interest. On the other hand, we also prove a new estimate for RBSDEs being sharper than that in El Karoui, Kapoudjian, Pardoux, Peng and Quenez [7], which turns out to be very useful because it allows to estimate the L^p -distance of the solutions of two different RBSDEs by the p -th power of the distance of the initial values of the driving forward equations. We also show that the unique viscosity solution of the approximating Isaacs equation which is constructed by the penalization method converges to the viscosity solution of the Isaacs equation with obstacle.

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1 Introduction

The general non-linear Backward Stochastic Differential Equations (BSDEs) were first introduced by Pardoux and Peng [16] in 1990. They have been studied since then by a lot of authors and have found various applications, namely in stochastic control, finance and the second order PDE theory. Fleming and Souganidis [9] studied in a rigorous manner two-player zero-sum stochastic differential games and proved that the lower and the upper value functions of such games satisfy the dynamic programming principle, that they are the unique viscosity solutions of the associated Bellman-Isaacs equations and coincide under the Isaacs condition. So a lot of recent works are based on the ideas developed in [9]; see, for instance, Buckdahn, Cardaliaguet and Rainer [2], Buckdahn and Li [3], Hou and Tang [13]. The reader interested in this subject is also referred to the references given in [9]. BSDE methods, originally developed by Peng [17], [18] for the stochastic control theory, have been introduced in the theory of stochastic differential games by Hamadène, Lepeltier [10] and Hamadène, Lepeltier and Peng [11] to study games with a dynamics whose diffusion coefficient is strictly elliptic and doesn't depend on the controls. In Buckdahn and Li [3] there isn't any such restriction on the diffusion coefficient and they used a completely new approach to study the stochastic differential games. In their framework the admissible controls can depend on the whole past, including information occurring before the beginning of the game, and, with the help of a Girsanov transformation argument, the a priori random lower and upper value functions were shown to be deterministic. This new approach in combination with BSDE methods (in particular the notion of stochastic backward semigroups, see Peng [17]) allowed them to prove the dynamic programming principle (DPP) for the upper and lower value functions of the game as well as to study the associated Isaacs equations in a very straight-forward way (i.e., in particular without making use of so called r -strategies and π -admissible strategies playing an essential role in [9]).

El Karoui, Kapoudjian, Pardoux, Peng and Quenez [7] first studied RBSDEs with one barrier. The solution of a RBSDE is a triplet (Y, Z, K) where a "reflection" forces the solution Y to stay above a given continuous stochastic process which is called "obstacle". This reflection is described by an increasing process K which pushes with minimal power the solution Y upwards the obstacle process S whenever it is touched by Y . The authors of [7] proved the existence and uniqueness of the solution by a fixed point argument as well as by approximation via penalization. They also studied the relation with the obstacle problem for nonlinear parabolic PDE's. In the Markov framework the solution Y of RBSDE provides a probabilistic formula for the unique viscosity solution of an obstacle problem for a parabolic partial differential equation. El Karoui, Pardoux and Quenez [8] found that the price process of an American option is the solution of an RBSDE. After that many authors have studied such equations and their applications.

Wu and Yu [19] studied a kind of stochastic recursive optimal control problem with obstacle constraints where the cost function is described by the solution of RBSDE. They used Peng's BSDE method in the control theory (Peng [17]) and require for this that all the coefficients are Lipschitz in their control variable. They show that the value function is the unique viscosity solution of an obstacle problem for the corresponding Hamilton-Jacobi-Bellman equations.

In this paper we investigate two-player zero-sum stochastic differential games. But different from the setting chosen by the papers mentioned above, we consider a more general running cost functional, which implies that the cost functionals will be given by doubly controlled RBSDEs.

They are interpreted as a payoff for Player I and as a cost for Player II and should exceed a given obstacle constraint. As usual in the differential game theory, the players cannot restrict to play only control processes, one player has to fix a strategy while the other player chooses the best answer to this strategy in form of a control process. The objective of our paper is to investigate these lower and upper value functions W and U (see (3.9) and (3.10)). The main results of the paper state that W and U are deterministic (Proposition 3.1) continuous viscosity solutions of the Bellman-Isaacs equations with obstacles (Theorem 4.1).

We emphasize that the fact that W and U , introduced as combination of essential infimum and essential supremum over a class of random variables, are deterministic is not trivial. For the proof of Proposition 3.1 we adapt the method from Buckdahn and Li [3]. This proposition then allows to prove the DPP (Theorem 3.1) in a straight forward way with the help of the method of stochastic backward semigroups introduced by Peng [17] and here extended to RBSDEs. However, we have to emphasize that the proof of the DPP for the stochastic differential games with reflection becomes more technical than that without reflection. One of the new elements of the proof of the DPP is an approved version of former estimates for RBSDE stated by El Karoui, Kapoudjian, Pardoux, Peng and Quenez [7] and by Wu and Yu [19]. In fact, we prove that, in the Markovian framework and under standard assumptions, the dependence of the solution on the random initial value of the driving SDE (on which also the obstacle process depends) is Lipschitz (Proposition 6.1), and not only Hölder with coefficient $1/2$. This improvement of the estimate is not only crucial for the proof of the DPP but has also its own interest.

We also underline that the proof that the lower and upper value functions W and U are deterministic (Proposition 3.1) continuous viscosity solutions of the associated Bellman-Isaacs equations with obstacles (Theorem 4.1) uses an argument which differs heavily from that used in [3] for the corresponding result without obstacle. In fact, we use the penalization method, and the proof that W is a viscosity subsolution (Proposition 4.2) turns out to be particularly complicated: it is based on a non evident translation of Peng's BSDE method to stochastic differential games with obstacle (see, in particular, Lemma 4.5). As a byproduct of our results we obtain that the viscosity solution of the penalized equation (4.5) converges to the viscosity solution of the Hamilton-Jacobi-Bellman-Isaacs equation with obstacle (4.1) (Theorem 4.2). Finally, we prove the uniqueness (Theorem 5.1) of the viscosity solutions W and U in a class of continuous functions with a growth condition which was introduced by Barles, Buckdahn and Pardoux [1], that is weaker than the polynomial growth assumption. Their proof has to be adapted to our framework (Lemma 5.1) because we don't have the continuity of the viscosity sub- and supersolution a priori but get it only by identifying both after the proof of the comparison principle (Theorem 5.1). In addition to the adaption of the proof to our framework we also simplify it considerably by reducing it to the comparison principle of Hamilton-Jacobi-Bellman equations.

Finally, let us point out that a work on stochastic differential games with two reflecting obstacles ([4]), which is based on the present paper, is available online. That work, separated from the present one in order to make the whole less heavy, has been used as a central key by Hamadène, Rotenstein and Zalinescu in their very recent paper [12].

Our paper is organized as follows. The Sections 2 and 6 recall some elements of the theory of BSDEs, RBSDEs and, in the Markovian framework, RBSDEs associated with forward SDEs, which will be needed in the sequel. Section 3 introduces the setting of stochastic differential games with

reflection and their lower and upper value functions W and U , and proves that these both functions are deterministic and satisfy the DPP. The DPP allows to prove that W and U are continuous. In Section 4 we prove that W and U are viscosity solutions of the associated Bellman-Isaacs equations with obstacles; the uniqueness is studied in Section 5. Finally, after having characterized W and U as the unique viscosity solutions of the associated Bellman-Isaacs equations with obstacles we show that W is less than or equal to U , and under the Isaacs condition, W and U coincide (one says that the game has a value). For the sake of readability of the paper the recall of basic properties of RBSDEs associated with forward SDEs, which are needed for our studies, is postponed to the appendix (Section 6). Some new results on RBSDEs are there given as well, namely Proposition 6.1, already announced above. Moreover, in the second part of the appendix (Section 7) we give for the reader's convenience the proofs of Proposition 3.1 and Theorem 3.1.

2 Preliminaries

The purpose of this section is to introduce some basic notions and results concerning backward and reflected backward SDEs, which will be needed in the subsequent sections. In all that follows we will work on the classical Wiener space (Ω, \mathcal{F}, P) : For an arbitrarily fixed time horizon $T > 0$, Ω is the set of all continuous functions from $[0, T]$ to \mathbb{R}^d , with initial value 0 ($\Omega = C_0([0, T]; \mathbb{R}^d)$) and \mathcal{F} is the Borel σ -algebra over Ω , completed by the Wiener measure on P . On this probability space the coordinate process $B_s(\omega) = \omega_s$, $s \in [0, T]$, $\omega \in \Omega$, is a d -dimensional Brownian motion. By $\mathbb{F} = \{\mathcal{F}_s, 0 \leq s \leq T\}$ we denote the natural filtration generated by the coordinate process B and augmented by all P -null sets, i.e.,

$$\mathcal{F}_s = \sigma\{B_r, r \leq s\} \vee \mathcal{N}_P, \quad s \in [0, T].$$

Here \mathcal{N}_P is the set of all P -null sets.

We introduce the following both spaces of processes which will be used frequently in the sequel:

$$\begin{aligned} \mathcal{S}^2(0, T; \mathbb{R}) &:= \{(\psi_t)_{0 \leq t \leq T} \text{ real-valued adapted càdlàg process :} \\ &\quad E[\sup_{0 \leq t \leq T} |\psi_t|^2] < +\infty\}; \end{aligned}$$

$$\begin{aligned} \mathcal{H}^2(0, T; \mathbb{R}^n) &:= \{(\psi_t)_{0 \leq t \leq T} \text{ } \mathbb{R}^n\text{-valued progressively measurable process :} \\ &\quad \|\psi\|_2^2 = E[\int_0^T |\psi_t|^2 dt] < +\infty\} \end{aligned}$$

(Recall that $|z|$ denotes the Euclidean norm of $z \in \mathbb{R}^n$). Given a measurable function $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ with the property that $(g(t, y, z))_{t \in [0, T]}$ is \mathbb{F} -progressively measurable for all (y, z) in $\mathbb{R} \times \mathbb{R}^d$, we make the following standard assumptions on g throughout the paper:

- (A1) There is some real $C \geq 0$ such that, P -a.s., for all $t \in [0, T]$, $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$,
$$|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|).$$
- (A2) $g(\cdot, 0, 0) \in \mathcal{H}^2(0, T; \mathbb{R})$.

The following result on BSDEs is by now well known, for its proof the reader is referred to the pioneering paper by Pardoux and Peng [16].

Lemma 2.1. *Let the function g satisfy the assumptions (A1) and (A2). Then, for any random variable $\xi \in L^2(\mathcal{O}, \mathcal{F}_T, P)$, the BSDE*

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T, \quad (2.1)$$

has a unique adapted solution

$$(Y_t, Z_t)_{t \in [0, T]} \in \mathcal{S}^2(0, T; \mathbb{R}) \times \mathcal{H}^2(0, T; \mathbb{R}^d).$$

In the sequel, we always assume that the driving coefficient g of a BSDE satisfies (A1) and (A2). Besides the existence and uniqueness result we shall also recall the comparison theorem for BSDEs (see Theorem 2.2 in El Karoui, Peng, Quenez [6] or Proposition 2.4 in Peng [18]).

Lemma 2.2. *(Comparison Theorem) Given two coefficients g_1 and g_2 satisfying (A1) and (A2) and two terminal values $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, P)$, we denote by (Y^1, Z^1) and (Y^2, Z^2) the solution of the BSDE with the data (ξ_1, g_1) and (ξ_2, g_2) , respectively. Then we have:*

- (i) *(Monotonicity) If $\xi_1 \geq \xi_2$ and $g_1 \geq g_2$, a.s., then $Y_t^1 \geq Y_t^2$, for all $t \in [0, T]$, a.s.*
- (ii) *(Strict Monotonicity) If, in addition to (i), we also assume that $P\{\xi_1 > \xi_2\} > 0$, then $P\{Y_t^1 > Y_t^2\} > 0$, for all $0 \leq t \leq T$, and in particular, $Y_0^1 > Y_0^2$.*

After this short and very basic recall on BSDEs let us consider now RBSDEs. An RBSDE is associated with a terminal condition $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, a generator g and an “obstacle” process $\{S_t\}_{0 \leq t \leq T}$. We shall make the following assumption on the obstacle process:

(A3) $\{S_t\}_{0 \leq t \leq T}$ is a continuous process such that $\{S_t\}_{0 \leq t \leq T} \in \mathcal{S}^2(0, T; \mathbb{R})$.

A solution of an RBSDE is a triplet (Y, Z, K) of \mathbb{F} -progressively measurable processes, taking its values in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+$ and satisfying the following properties

- (i) $Y \in \mathcal{S}^2(0, T; \mathbb{R})$, $Z \in \mathcal{H}^2(0, T; \mathbb{R}^d)$ and $K_T \in L^2(\Omega, \mathcal{F}_T, P)$;
- (ii) $Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s, \quad t \in [0, T];$ (2.2)
- (iii) $Y_t \geq S_t$, a.s., for any $t \in [0, T]$;
- (iv) $\{K_t\}$ is continuous and increasing, $K_0 = 0$ and $\int_0^T (Y_t - S_t) dK_t = 0$.

The following two lemmas are borrowed from Theorem 5.2 and Theorem 4.1, respectively, of the paper [7] by El Karoui, Kapoudjian, Pardoux, Peng and Quenez.

Lemma 2.3. *Assume that g satisfies (A1) and (A2), ξ belongs to $L^2(\Omega, \mathcal{F}_T, P)$, $\{S_t\}_{0 \leq t \leq T}$ satisfies (A3), and $S_T \leq \xi$ a.s. Then RBSDE (2.2) has a unique solution (Y, Z, K) .*

Remark 2.1. *For shortness, a given triplet (ξ, g, S) is said to satisfy the Standard Assumptions if the generator g satisfies (A1) and (A2), the terminal value ξ belongs to $L^2(\Omega, \mathcal{F}_T, P)$, the obstacle process S satisfies (A3) and $S_T \leq \xi$, a.s.*

Lemma 2.4. (*Comparison Theorem*) We suppose that two triplets (ξ_1, g_1, S^1) and (ξ_2, g_2, S^2) satisfy the Standard Assumptions but assume only for one of the both coefficients g_1 and g_2 to be Lipschitz. Furthermore, we make the following assumptions:

- (i) $\xi_1 \leq \xi_2$, a.s.;
- (ii) $g_1(t, y, z) \leq g_2(t, y, z)$, a.s., for $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$;
- (iii) $S_t^1 \leq S_t^2$, a.s., for $t \in [0, T]$.

Let (Y^1, Z^1, K^1) and (Y^2, Z^2, K^2) be adapted solutions of RBSDEs (2.2) with data (ξ_1, g_1, S^1) and (ξ_2, g_2, S^2) , respectively. Then, $Y_t^1 \leq Y_t^2$, a.s., for $t \in [0, T]$.

We will also need the following standard results on RBSDEs.

Lemma 2.5. Let (Y, Z, K) be the solution of the above RBSDE (2.2) with data (ξ, g, S) satisfying the Standard Assumptions. Then, there exists a constant C such that

$$E\left[\sup_{t \leq s \leq T} |Y_s|^2 + \int_t^T |Z_s|^2 ds + |K_T - K_t|^2 | \mathcal{F}_t\right] \leq CE[\xi^2 + \left(\int_t^T g(s, 0, 0) ds\right)^2] + \sup_{t \leq s \leq T} S_s^2 | \mathcal{F}_t\right].$$

The constant C depends only on the Lipschitz constant of g .

Lemma 2.6. Let (ξ, g, S) and (ξ', g', S') be two triplets satisfying the above Standard Assumptions. We suppose that (Y, Z, K) and (Y', Z', K') are the solutions of RBSDE (2.2) with the data (ξ, g, S) and (ξ', g', S') , respectively. Then there exists a constant C such that, with the notations,

$$\begin{aligned} \Delta \xi &= \xi - \xi', & \Delta g &= g - g', & \Delta S &= S - S'; \\ \Delta Y &= Y - Y', & \Delta Z &= Z - Z', & \Delta K &= K - K', \end{aligned}$$

it holds

$$\begin{aligned} &E\left[\sup_{t \leq s \leq T} |\Delta Y_s|^2 + \int_t^T |\Delta Z_s|^2 ds + |\Delta K_T - \Delta K_t|^2 | \mathcal{F}_t\right] \\ &\leq CE[|\Delta \xi|^2 + \left(\int_t^T |\Delta g(s, Y_s, Z_s)| ds\right)^2 | \mathcal{F}_t] + C \left(E\left[\sup_{t \leq s \leq T} |\Delta S_s|^2 | \mathcal{F}_t\right]\right)^{1/2} \Psi_{t,T}^{1/2}, \end{aligned}$$

where

$$\begin{aligned} \Psi_{t,T} &= E[|\xi|^2 + \left(\int_t^T |g(s, 0, 0)| ds\right)^2] + \sup_{t \leq s \leq T} |S_s|^2 \\ &\quad + |\xi'|^2 + \left(\int_t^T |g'(s, 0, 0)| ds\right)^2 + \sup_{t \leq s \leq T} |S'_s|^2 | \mathcal{F}_t|. \end{aligned}$$

The constant C depends only on the Lipschitz constant of g' .

The Lemmas 2.5 and 2.6 are based on the Propositions 3.5 and 3.6 in [7] and their generalizations by the Propositions 2.1 and 2.2 in [19], respectively.

Remark 2.2. For the Markovian situation in which the obstacle process is a deterministic function, we can improve Lemma 2.6 considerably and show that Y is Lipschitz continuous with respect to the possibly random initial condition of the driving SDE (whose solution intervenes in the RBSDEs as well as in the obstacles), see Proposition 6.1 in the Section 6.

3 Stochastic Differential Games with Reflections and Associated Dynamic Programming Principles

We now introduce the framework of our study of stochastic differential games with reflection for two players. We will denote the control state space of the first player by U , and that of the second one by V ; the associated sets of admissible controls will be denoted by \mathcal{U} and \mathcal{V} , respectively. The set \mathcal{U} is formed by all U -valued \mathbb{F} -progressively measurable processes and \mathcal{V} is the set of all V -valued \mathbb{F} -progressively measurable processes. The control state spaces U and V are supposed to be compact metric spaces.

For given admissible controls $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, the according orbit which regards t as the initial time and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$ as the initial state is defined by the solution of the following SDE:

$$\begin{cases} dX_s^{t,\zeta;u,v} &= b(s, X_s^{t,\zeta;u,v}, u_s, v_s)ds + \sigma(s, X_s^{t,\zeta;u,v}, u_s, v_s)dB_s, \quad s \in [t, T], \\ X_t^{t,\zeta;u,v} &= \zeta, \end{cases} \quad (3.1)$$

where the mappings

$$b : [0, T] \times \mathbb{R}^n \times U \times V \rightarrow \mathbb{R}^n \text{ and } \sigma : [0, T] \times \mathbb{R}^n \times U \times V \rightarrow \mathbb{R}^{n \times d}$$

satisfy the following conditions:

- (i) For every fixed $x \in \mathbb{R}^n$, $b(\cdot, x, \cdot, \cdot)$ and $\sigma(\cdot, x, \cdot, \cdot)$ are continuous in (t, u, v) ;
 - (ii) There exists a $C > 0$ such that, for all $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, $u \in U$, $v \in V$, $|b(t, x, u, v) - b(t, x', u, v)| + |\sigma(t, x, u, v) - \sigma(t, x', u, v)| \leq C|x - x'|$.
- (H3.1)

From (H3.1) we can get the global linear growth conditions of b and σ , i.e., the existence of some $C > 0$ such that, for all $0 \leq t \leq T$, $u \in U$, $v \in V$, $x \in \mathbb{R}^n$,

$$|b(t, x, u, v)| + |\sigma(t, x, u, v)| \leq C(1 + |x|). \quad (3.2)$$

As recalled in Section 6, (6.2), it follows that, under the above assumptions, for any $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, SDE (3.1) has a unique strong solution. Moreover, for any $p \geq 2$, there exists $C_p \in \mathbb{R}$ such that, for any $t \in [0, T]$, $u(\cdot) \in \mathcal{U}$, $v(\cdot) \in \mathcal{V}$ and $\zeta, \zeta' \in L^p(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we also have the following estimates, P-a.s.:

$$\begin{aligned} E\left[\sup_{s \in [t, T]} |X_s^{t,\zeta;u,v} - X_s^{t,\zeta';u,v}|^p \middle| \mathcal{F}_t\right] &\leq C_p |\zeta - \zeta'|^p, \\ E\left[\sup_{s \in [t, T]} |X_s^{t,\zeta;u,v}|^p \middle| \mathcal{F}_t\right] &\leq C_p (1 + |\zeta|^p). \end{aligned} \quad (3.3)$$

The constant C_p depends only on the Lipschitz and the linear growth constants of b and σ with respect to x .

Let now be given three functions

$$\Phi : \mathbb{R}^n \rightarrow \mathbb{R}, \quad h : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}, \quad f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times U \times V \rightarrow \mathbb{R}$$

that satisfy the following conditions:

- (i) For every fixed $(x, y, z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$, $f(\cdot, x, y, z, \cdot, \cdot)$ is continuous in (t, u, v) and there exists a constant $C > 0$ such that, for all $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, $u \in U$ and $v \in V$,
$$\begin{aligned} |f(t, x, y, z, u, v) - f(t, x', y', z', u, v)| \\ \leq C(|x - x'| + |y - y'| + |z - z'|); \end{aligned}$$
- (ii) There is a constant $C > 0$ such that, for all $x, x' \in \mathbb{R}^n$,
$$|\Phi(x) - \Phi(x')| \leq C|x - x'|;$$
- (iii) For every fixed $x \in \mathbb{R}^n$, $h(\cdot, x)$ is continuous in t and there is a constant $C > 0$ such that, for all $t \in [0, T]$, $x, x' \in \mathbb{R}^n$,
$$|h(t, x) - h(t, x')| \leq C|x - x'|. \quad (\text{H3.2})$$

Moreover,

$$h(T, x) \leq \Phi(x), \text{ for all } x \in \mathbb{R}^n.$$

From (H3.2) we see that f , h and Φ also satisfy the global linear growth condition in x , i.e., there exists some $C > 0$ such that, for all $0 \leq t \leq T$, $u \in U$, $v \in V$, $x \in \mathbb{R}^n$,

$$|f(t, x, 0, 0, u, v)| + |\Phi(x)| + |h(t, x)| \leq C(1 + |x|). \quad (3.4)$$

Let $t \in [0, T]$. For any $u(\cdot) \in \mathcal{U}$, $v(\cdot) \in \mathcal{V}$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, the mappings $\xi := \Phi(X_T^{t, \zeta; u, v})$, $S_s = h(s, X_s^{t, \zeta; u, v})$ and $g(s, y, z) := f(s, X_s^{t, \zeta; u, v}, y, z, u_s, v_s)$ satisfy the conditions of Lemma 2.3 on the interval $[t, T]$. Therefore, there exists a unique solution to the following RBSDE:

$$\begin{aligned} & \text{(i)} Y^{t, \zeta; u, v} \in \mathcal{S}^2(t, T; \mathbb{R}), \quad Z^{t, \zeta; u, v} \in \mathcal{H}^2(t, T; \mathbb{R}^d), \text{ and } K_T^{t, \zeta; u, v} \in L^2(\Omega, \mathcal{F}_T, P); \\ & \text{(ii)} Y_s^{t, \zeta; u, v} = \Phi(X_T^{t, \zeta; u, v}) + \int_s^T f(r, X_r^{t, \zeta; u, v}, Y_r^{t, \zeta; u, v}, Z_r^{t, \zeta; u, v}, u_r, v_r) dr + K_T^{t, \zeta; u, v} \\ & \quad - K_s^{t, \zeta; u, v} - \int_s^T Z_r^{t, \zeta; u, v} dB_r, \quad s \in [t, T]; \\ & \text{(iii)} Y_s^{t, \zeta; u, v} \geq h(s, X_s^{t, \zeta; u, v}), \quad \text{a.s., for any } s \in [t, T]; \\ & \text{(iv)} K^{t, \zeta; u, v} \text{ is continuous and increasing, } K_t^{t, \zeta; u, v} = 0, \\ & \quad \int_t^T (Y_r^{t, \zeta; u, v} - h(r, X_r^{t, \zeta; u, v})) dK_r^{t, \zeta; u, v} = 0, \end{aligned} \quad (3.5)$$

where $X^{t, \zeta; u, v}$ is introduced by equation (3.1).

Moreover, in analogy to Proposition 6.1, we can see that there exists some constant $C > 0$ such that, for all $0 \leq t \leq T$, $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, P-a.s.,

$$\begin{aligned} & \text{(i)} \quad |Y_t^{t, \zeta; u, v} - Y_t^{t, \zeta'; u, v}| \leq C|\zeta - \zeta'|; \\ & \text{(ii)} \quad |Y_t^{t, \zeta; u, v}| \leq C(1 + |\zeta|). \end{aligned} \quad (3.6)$$

Now, similar to Buckdahn and Li [3], we introduce the following subspaces of admissible controls and the definitions of admissible strategies for the game:

Definition 3.1. An admissible control process $u = \{u_r, r \in [t, s]\}$ (resp., $v = \{v_r, r \in [t, s]\}$) for Player I (resp., II) on $[t, s]$ ($t < s \leq T$) is an \mathcal{F}_r -progressively measurable process taking values in U (resp., V). The set of all admissible controls for Player I (resp., II) on $[t, s]$ is denoted by $\mathcal{U}_{t,s}$ (resp., $\mathcal{V}_{t,s}$). We identify two processes u and \bar{u} in $\mathcal{U}_{t,s}$ and write $u \equiv \bar{u}$ on $[t, s]$, if $P\{u = \bar{u} \text{ a.e. in } [t, s]\} = 1$. Similarly, we interpret $v \equiv \bar{v}$ on $[t, s]$ for two elements v and \bar{v} of $\mathcal{V}_{t,s}$.

Definition 3.2. A nonanticipative strategy for Player I on $[t, s]$ ($t < s \leq T$) is a mapping $\alpha : \mathcal{V}_{t,s} \longrightarrow \mathcal{U}_{t,s}$ such that, for any \mathcal{F}_r -stopping time $S : \Omega \rightarrow [t, s]$ and any $v_1, v_2 \in \mathcal{V}_{t,s}$ with $v_1 \equiv v_2$ on $\llbracket t, S \rrbracket$, it holds $\alpha(v_1) \equiv \alpha(v_2)$ on $\llbracket t, S \rrbracket$. Nonanticipative strategies for Player II on $[t, s]$, $\beta : \mathcal{U}_{t,s} \longrightarrow \mathcal{V}_{t,s}$, are defined similarly. The set of all nonanticipative strategies $\alpha : \mathcal{V}_{t,s} \longrightarrow \mathcal{U}_{t,s}$ for Player I on $[t, s]$ is denoted by $\mathcal{A}_{t,s}$. The set of all nonanticipative strategies $\beta : \mathcal{U}_{t,s} \longrightarrow \mathcal{V}_{t,s}$ for Player II on $[t, s]$ is denoted by $\mathcal{B}_{t,s}$. (Recall that $\llbracket t, S \rrbracket = \{(r, \omega) \in [0, T] \times \Omega, t \leq r \leq S(\omega)\}$.)

Given the control processes $u(\cdot) \in \mathcal{U}_{t,T}$ and $v(\cdot) \in \mathcal{V}_{t,T}$ we introduce the following associated cost functional

$$J(t, x; u, v) := Y_t^{t,x;u,v}, \quad (t, x) \in [0, T] \times \mathbb{R}^n, \quad (3.7)$$

where the process $Y^{t,x;u,v}$ is defined by RBSDE (3.5).

Similarly to the proof of Proposition 6.2 we can get that, for any $t \in [0, T]$, $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$,

$$J(t, \zeta; u, v) = Y_t^{t,\zeta;u,v}, \quad P\text{-a.s.} \quad (3.8)$$

We emphasize that $J(t, \zeta; u, v) = J(t, x; u, v)|_{x=\zeta}$ while $Y^{t,\zeta;u,v}$ is defined by (3.5). Being particularly interested in the case of a deterministic ζ , i.e., $\zeta = x \in \mathbb{R}^n$, we define the lower value function of our stochastic differential game with reflection

$$W(t, x) := \operatorname{essinf}_{\beta \in \mathcal{B}_{t,T}} \operatorname{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u, \beta(u)) \quad (3.9)$$

and its upper value function

$$U(t, x) := \operatorname{esssup}_{\alpha \in \mathcal{A}_{t,T}} \operatorname{essinf}_{v \in \mathcal{V}_{t,T}} J(t, x; \alpha(v), v). \quad (3.10)$$

The names “lower value function” and “upper value function” for W and U , respectively, are justified later by Remark 5.1.

Remark 3.1. Obviously, under the assumptions (H3.1)-(H3.2), the lower value function $W(t, x)$ as well as the upper value function $U(t, x)$ are well-defined and a priori they both are bounded \mathcal{F}_t -measurable random variables. But it turns out that $W(t, x)$ and $U(t, x)$ are even deterministic. For proving this we adapt the new approach by Buckdahn and Li [3]. In the sequel we will concentrate on the study of the properties of W . The discussion of the properties of U which are comparable with those of W can be carried out in a similar manner.

Proposition 3.1. For any $(t, x) \in [0, T] \times \mathbb{R}^n$, we have $W(t, x) = E[W(t, x)]$, P -a.s. Identifying $W(t, x)$ with its deterministic version $E[W(t, x)]$ we can consider $W : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ as a deterministic function.

The proof of Proposition 3.1 is similar to that of Proposition 3.3 in [3]. However, for the reader’s convenience we give the proof in the Appendix II.

The first property of the lower value function $W(t, x)$ which we present below is an immediate consequence of (3.6) and (3.9).

Lemma 3.1. There exists a constant $C > 0$ such that, for all $0 \leq t \leq T$, $x, x' \in \mathbb{R}^n$,

$$\begin{aligned} \text{(i)} \quad & |W(t, x) - W(t, x')| \leq C|x - x'|; \\ \text{(ii)} \quad & |W(t, x)| \leq C(1 + |x|). \end{aligned} \quad (3.11)$$

□

We now discuss (the generalized) DPP for our stochastic differential game with reflection (3.1), (3.5) and (3.9). For this end we have to define the family of (backward) semigroups associated with RBSDE (3.5). This notion of stochastic backward semigroups was first introduced by Peng [17] and applied to study the DPP for stochastic control problems. Our approach adapts Peng's ideas to the framework of stochastic differential games with reflection.

Given the initial data (t, x) , a positive number $\delta \leq T - t$, admissible control processes $u(\cdot) \in \mathcal{U}_{t,t+\delta}$, $v(\cdot) \in \mathcal{V}_{t,t+\delta}$ and a real-valued random variable $\eta \in L^2(\Omega, \mathcal{F}_{t+\delta}, P; \mathbb{R})$ such that $\eta \geq h(t + \delta, X_{t+\delta}^{t,x;u,v})$, a.s., we put

$$G_{s,t+\delta}^{t,x;u,v}[\eta] := \tilde{Y}_s^{t,x;u,v}, \quad s \in [t, t + \delta], \quad (3.12)$$

where the triplet $(\tilde{Y}_s^{t,x;u,v}, \tilde{Z}_s^{t,x;u,v}, \tilde{K}_s^{t,x;u,v})_{t \leq s \leq t+\delta}$ is the solution of the following RBSDE with time horizon $t + \delta$:

$$\begin{aligned} & \text{(i)} \tilde{Y}_s^{t,x;u,v} \in \mathcal{S}^2(t, t + \delta; \mathbb{R}), \tilde{Z}_s^{t,x;u,v} \in \mathcal{H}^2(t, t + \delta; \mathbb{R}^d), \text{ and } \tilde{K}_{t+\delta}^{t,x;u,v} \in L^2(\Omega, \mathcal{F}_{t+\delta}, P); \\ & \text{(ii)} \tilde{Y}_s^{t,x;u,v} = \eta + \int_s^{t+\delta} f(r, X_r^{t,x;u,v}, \tilde{Y}_r^{t,x;u,v}, \tilde{Z}_r^{t,x;u,v}, u_r, v_r) dr + \tilde{K}_{t+\delta}^{t,x;u,v} \\ & \quad - \tilde{K}_s^{t,x;u,v} - \int_s^{t+\delta} \tilde{Z}_r^{t,x;u,v} dB_r, \quad s \in [t, t + \delta]; \\ & \text{(iii)} \tilde{Y}_s^{t,x;u,v} \geq h(s, X_s^{t,x;u,v}), \text{ a.s., for any } s \in [t, t + \delta]; \\ & \text{(iv)} \tilde{K}_t^{t,x;u,v} = 0, \quad \int_t^{t+\delta} (\tilde{Y}_r^{t,x;u,v} - h(r, X_r^{t,x;u,v})) d\tilde{K}_r^{t,x;u,v} = 0, \end{aligned} \quad (3.13)$$

where $X^{t,x;u,v}$ is introduced by equation (3.1).

Then, in particular, for the solution $(Y^{t,x;u,v}, Z^{t,x;u,v}, K^{t,x;u,v})$ of RBSDE (3.5) we have

$$G_{t,T}^{t,x;u,v}[\Phi(X_T^{t,x;u,v})] = G_{t,t+\delta}^{t,x;u,v}[Y_{t+\delta}^{t,x;u,v}]. \quad (3.14)$$

Moreover,

$$\begin{aligned} J(t, x; u, v) &= Y_t^{t,x;u,v} = G_{t,T}^{t,x;u,v}[\Phi(X_T^{t,x;u,v})] = G_{t,t+\delta}^{t,x;u,v}[Y_{t+\delta}^{t,x;u,v}] \\ &= G_{t,t+\delta}^{t,x;u,v}[J(t + \delta, X_{t+\delta}^{t,x;u,v}; u, v)]. \end{aligned}$$

Remark 3.2. For the better comprehension of the reader let us point out that if f is independent of (y, z) then

$$G_{s,t+\delta}^{t,x;u,v}[\eta] = E[\eta + \int_s^{t+\delta} f(r, X_r^{t,x;u,v}, u_r, v_r) dr + \tilde{K}_{t+\delta}^{t,x;u,v} - \tilde{K}_s^{t,x;u,v} | \mathcal{F}_s], \quad s \in [t, t + \delta].$$

Theorem 3.1. Under the assumptions (H3.1) and (H3.2), the lower value function $W(t, x)$ obeys the following DPP : For any $0 \leq t < t + \delta \leq T$, $x \in \mathbb{R}^n$,

$$W(t, x) = \text{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \text{esssup}_{u \in \mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{t,x;u,\beta(u)}[W(t + \delta, X_{t+\delta}^{t,x;u,\beta(u)})]. \quad (3.15)$$

The proof of Theorem 3.1 is very technique. But because we have got Proposition 6.1 the proof becomes possible with the help of the method of BSDE. On the other hand, we should pay attention to make sure the terminal condition is always above the obstacle. For the reader's convenience we give the proof in the Appendix II.

Remark 3.3. (i) From the proof of Theorem 3.1 (inequalities (7.2) and (7.7)) we see that, for all $(t, x) \in [0, T] \times \mathbb{R}^n$, $\delta > 0$ with $0 < \delta \leq T - t$ and $\varepsilon > 0$, it holds:

a) For every $\beta \in \mathcal{B}_{t,t+\delta}$, there exists some $u^\varepsilon(\cdot) \in \mathcal{U}_{t,t+\delta}$ such that

$$W(t, x) (= W_\delta(t, x)) \leq G_{t,t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}[W(t + \delta, X_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)})] + \varepsilon, \quad P\text{-a.s.} \quad (3.16)$$

b) There exists some $\beta^\varepsilon \in \mathcal{B}_{t,t+\delta}$ such that, for all $u \in \mathcal{U}_{t,t+\delta}$,

$$W(t, x) (= W_\delta(t, x)) \geq G_{t,t+\delta}^{t,x;u,\beta^\varepsilon(u)}[W(t + \delta, X_{t+\delta}^{t,x;u,\beta^\varepsilon(u)})] - \varepsilon, \quad P\text{-a.s.} \quad (3.17)$$

(ii) Recall that the lower value function W is deterministic. Thus, for $\delta = T - t$, by taking the expectation on both sides of (3.16) and (3.17) we can show that

$$W(t, x) = \inf_{\beta \in \mathcal{B}_{t,T}} \sup_{u \in \mathcal{U}_{t,T}} E[J(t, x; u, \beta(u))].$$

For this we recall that

$$W(T, X_T^{t,x;u,\beta(u)}) = \Phi(X_T^{t,x;u,\beta(u)}).$$

In analogy we also have

$$U(t, x) = \sup_{\alpha \in \mathcal{A}_{t,T}} \inf_{v \in \mathcal{V}_{t,T}} E[J(t, x; \alpha(v), v)].$$

In Lemma 3.2 we have already seen that the lower value function $W(t, x)$ is Lipschitz continuous in x , uniformly in t . With the help of Theorem 3.1 we can now also study the continuity of $W(t, x)$ in t .

Theorem 3.2. Let us suppose that the assumptions (H3.1) and (H3.2) hold. Then the lower value function $W(t, x)$ is continuous in t .

Proof. Let $(t, x) \in [0, T] \times \mathbb{R}^n$ and $\delta > 0$ be arbitrarily given such that $0 < \delta \leq T - t$. Our objective is to prove the following inequality by using (3.16) and (3.17):

$$\begin{aligned} & -C(1 + |x|)\delta^{\frac{1}{2}} - C(1 + |x|^{\frac{1}{2}})\delta^{\frac{1}{4}} - C \sup_{t \leq s \leq t+\delta} |h(s, x) - h(t + \delta, x)|^{\frac{1}{2}} \\ & \leq W(t, x) - W(t + \delta, x) \\ & \leq C(1 + |x|)\delta^{\frac{1}{2}} + C(1 + |x|^{\frac{1}{2}})\delta^{\frac{1}{4}} + C \sup_{t \leq s \leq t+\delta} |h(s, x) - h(t + \delta, x)|^{\frac{1}{2}}. \end{aligned} \quad (3.18)$$

From here we obtain immediately that W is continuous in t . We will only check the second inequality in (3.18), the first one can be shown in a similar way. To this end we note that due to (3.16), for an arbitrarily small $\varepsilon > 0$,

$$W(t, x) - W(t + \delta, x) \leq I_\delta^1 + I_\delta^2 + \varepsilon, \quad (3.19)$$

where

$$\begin{aligned} I_\delta^1 &:= G_{t,t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}[W(t + \delta, X_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)})] - G_{t,t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}[W(t + \delta, x)], \\ I_\delta^2 &:= G_{t,t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}[W(t + \delta, x)] - W(t + \delta, x), \end{aligned}$$

for arbitrarily chosen $\beta \in \mathcal{B}_{t,t+\delta}$ and $u^\varepsilon \in \mathcal{U}_{t,t+\delta}$ such that (3.16) holds. From Lemma 2.6 and the estimate (3.11) we obtain that, for some constant C independent of the controls u^ε and $\beta(u^\varepsilon)$,

$$\begin{aligned} |I_\delta^1| &\leq [CE(|W(t + \delta, X_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}) - W(t + \delta, x)|^2 | \mathcal{F}_t)]^{\frac{1}{2}} \\ &\leq [CE(|X_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)} - x|^2 | \mathcal{F}_t)]^{\frac{1}{2}}, \end{aligned}$$

and since $E[|X_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)} - x|^2|\mathcal{F}_t] \leq C(1+|x|^2)\delta$ we deduce that $|I_\delta^1| \leq C(1+|x|)\delta^{\frac{1}{2}}$. Note that $W(t+\delta, x) \geq h(t+\delta, x)$. Then $(Y, Z, K) = (W(t+\delta, x), 0, 0)$ is the solution of RBSDE (2.2) on the interval $[t, t+\delta]$ with the data $\zeta = W(t+\delta, x), g \equiv 0, S_s = h(t+\delta, x)$. On the other hand, from the definition of $G_{t,t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}[\cdot]$ (see (3.12)) and Lemma 2.6 we know that the second term I_δ^2 can be estimated by

$$\begin{aligned} |I_\delta^2|^2 &\leq E[(\int_t^{t+\delta} f(s, X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}, W(t+\delta, x), 0, u_s^\varepsilon, \beta_s(u^\varepsilon)) ds)^2|\mathcal{F}_t] \\ &\quad + C(E[\sup_{t \leq s \leq t+\delta} |h(s, X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}) - h(t+\delta, x)|^2|\mathcal{F}_t])^{\frac{1}{2}} \\ &=: I_{3,\delta} + I_{4,\delta}, \end{aligned}$$

where, by Schwartz inequality as well as the estimates (3.3) and (3.11),

$$\begin{aligned} |I_{3,\delta}|^{\frac{1}{2}} &\leq \delta^{\frac{1}{2}} E[\int_t^{t+\delta} |f(s, X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}, W(t+\delta, x), 0, u_s^\varepsilon, \beta_s(u^\varepsilon))|^2 ds|\mathcal{F}_t]^{\frac{1}{2}} \\ &\leq \delta^{\frac{1}{2}} E[\int_t^{t+\delta} (|f(s, X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}, 0, 0, u_s^\varepsilon, \beta_s(u^\varepsilon))| + C|W(t+\delta, x)|)^2 ds|\mathcal{F}_t]^{\frac{1}{2}} \\ &\leq C\delta^{\frac{1}{2}} E[\int_t^{t+\delta} (|1 + |X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}| + |W(t+\delta, x)|)^2 ds|\mathcal{F}_t]^{\frac{1}{2}} \\ &\leq C(1+|x|)\delta \end{aligned}$$

and

$$\begin{aligned} |I_{4,\delta}|^2 &\leq CE[\sup_{t \leq s \leq t+\delta} |h(s, X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}) - h(s, x) + h(s, x) - h(t+\delta, x)|^2|\mathcal{F}_t] \\ &\leq CE[\sup_{t \leq s \leq t+\delta} |h(s, X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}) - h(s, x)|^2|\mathcal{F}_t] + C[\sup_{t \leq s \leq t+\delta} |h(s, x) - h(t+\delta, x)|^2] \\ &\leq CE[\sup_{t \leq s \leq t+\delta} |X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)} - x|^2|\mathcal{F}_t] + C[\sup_{t \leq s \leq t+\delta} |h(s, x) - h(t+\delta, x)|^2] \\ &\leq C(1+|x|^2)\delta + C[\sup_{t \leq s \leq t+\delta} |h(s, x) - h(t+\delta, x)|^2]. \end{aligned}$$

Hence, from (3.19) and letting $\varepsilon \downarrow 0$ we get the second inequality of (3.18). The proof is complete. \square

4 Viscosity Solution of Isaacs Equation with Obstacle: Existence Theorem

In this section we consider the following Isaacs equations with obstacles

$$\begin{cases} \min\{W(t, x) - h(t, x), -\frac{\partial}{\partial t}W(t, x) - H^-(t, x, W, DW, D^2W)\} = 0, & (t, x) \in [0, T] \times \mathbb{R}^n, \\ W(T, x) = \Phi(x), & x \in \mathbb{R}^n, \end{cases} \quad (4.1)$$

and

$$\begin{cases} \min\{U(t, x) - h(t, x), -\frac{\partial}{\partial t}U(t, x) - H^+(t, x, U, DU, D^2U)\} = 0, & (t, x) \in [0, T] \times \mathbb{R}^n, \\ U(T, x) = \Phi(x), & x \in \mathbb{R}^n, \end{cases} \quad (4.2)$$

associated with the Hamiltonians

$$H^-(t, x, y, q, X) = \sup_{u \in U} \inf_{v \in V} \left\{ \frac{1}{2} \text{tr}(\sigma \sigma^T(t, x, u, v) X) + q \cdot b(t, x, u, v) + f(t, x, y, q \cdot \sigma, u, v) \right\}$$

and

$$H^+(t, x, y, q, X) = \inf_{v \in V} \sup_{u \in U} \left\{ \frac{1}{2} \text{tr}(\sigma \sigma^T(t, x, u, v) X) + q \cdot b(t, x, u, v) + f(t, x, y, q \cdot \sigma, u, v) \right\},$$

respectively, where $t \in [0, T]$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $q \in \mathbb{R}^n$ and $X \in \mathbb{S}^n$ (recall that \mathbb{S}^n denotes the set of $n \times n$ symmetric matrices). Here the functions b, σ, f and Φ are supposed to satisfy (H3.1) and (H3.2), respectively.

In this section we want to prove that the lower value function $W(t, x)$ introduced by (3.9) is the viscosity solution of equation (4.1), while the upper value function $U(t, x)$ defined by (3.10) is the viscosity solution of equation (4.2). The uniqueness of the viscosity solution will be shown in the next section for the class of continuous functions satisfying some growth assumption which is weaker than the polynomial growth condition. We first recall the definition of a viscosity solution of equation (4.1), that for equation (4.2) is similar. We borrow the definitions from Crandall, Ishii and Lions [5].

Definition 4.1. (i) A real-valued upper semicontinuous function $W : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is called a viscosity subsolution of equation (4.1) if $W(T, x) \leq \Phi(x)$, for all $x \in \mathbb{R}^n$, and if for all functions $\varphi \in C_{l,b}^3([0, T] \times \mathbb{R}^n)$ and $(t, x) \in [0, T] \times \mathbb{R}^n$ such that $W - \varphi$ attains its local maximum at (t, x) , we have

$$\min \left(W(t, x) - h(t, x), -\frac{\partial \varphi}{\partial t}(t, x) - H^-(t, x, W, D\varphi, D^2\varphi) \right) \leq 0; \quad (4.1')$$

(ii) A real-valued lower semicontinuous function $W : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is called a viscosity supersolution of equation (4.1) if $W(T, x) \geq \Phi(x)$, for all $x \in \mathbb{R}^n$, and if for all functions $\varphi \in C_{l,b}^3([0, T] \times \mathbb{R}^n)$ and $(t, x) \in [0, T] \times \mathbb{R}^n$ such that $W - \varphi$ attains its local minimum at (t, x) , it holds

$$\min \left(W(t, x) - h(t, x), -\frac{\partial \varphi}{\partial t}(t, x) - H^-(t, x, W, D\varphi, D^2\varphi) \right) \geq 0; \quad (4.1'')$$

(iii) A real-valued continuous function $W \in C([0, T] \times \mathbb{R}^n)$ is called a viscosity solution of equation (4.1) if it is both a viscosity sub- and a supersolution of equation (4.1).

Remark 4.1. $C_{l,b}^3([0, T] \times \mathbb{R}^n)$ denotes the set of the real-valued functions that are continuously differentiable up to the third order and whose derivatives of order from 1 to 3 are bounded.

We now state the main result of this section.

Theorem 4.1. Under the assumptions (H3.1) and (H3.2) the lower value function W defined by (3.9) is a viscosity solution of Isaacs equation (4.1), while U defined by (3.10) solves the Isaacs equation (4.2) in the viscosity solution sense.

We will develop the proof of this theorem only for W , that of U is analogous. The proof is mainly based on an approximation of our RBSDE (3.5) by a sequence of penalized BSDEs. This generalization method for RBSDEs was first studied in [9], Section 6 (pp.719-pp.723).

For each $(t, x) \in [0, T] \times \mathbb{R}^n$, and $m \in \mathbb{N}$, let $\{(^m Y_s^{t,x;u,v}, ^m Z_s^{t,x;u,v}), t \leq s \leq T\}$ denote the solution of the BSDE

$$\begin{aligned} ^m Y_s^{t,x;u,v} &= \Phi(X_T^{t,x;u,v}) + \int_s^T f(r, X_r^{t,x;u,v}, ^m Y_r^{t,x;u,v}, ^m Z_r^{t,x;u,v}, u_r, v_r) dr \\ &\quad + m \int_s^T (^m Y_r^{t,x;u,v} - h(r, X_r^{t,x;u,v}))^- dr - \int_s^T ^m Z_r^{t,x;u,v} dW_r, \quad t \leq s \leq T. \end{aligned}$$

We define

$$J_m(t, x; u, v) := ^m Y_t^{t,x;u,v}, \quad u \in \mathcal{U}_{t,T}, \quad v \in \mathcal{V}_{t,T}, \quad 0 \leq t \leq T, \quad x \in \mathbb{R}^n, \quad (4.3)$$

and consider the lower value function

$$W_m(t, x) := \text{essinf}_{\beta \in \mathcal{B}_{t,T}} \text{esssup}_{u \in \mathcal{U}_{t,T}} J_m(t, x; u, \beta(u)), \quad 0 \leq t \leq T, \quad x \in \mathbb{R}^n. \quad (4.4)$$

It is known from Buckdahn and Li [3] that $W_m(t, x)$ defined in (4.4) is in $C([0, T] \times \mathbb{R}^n)$, has linear growth in x , and is the unique continuous viscosity solution of the following Isaacs equations:

$$\begin{cases} -\frac{\partial}{\partial t} W_m(t, x) - \sup_{u \in U} \inf_{v \in V} \{ \frac{1}{2} \text{tr}(\sigma \sigma^T(t, x, u, v) D^2 W_m(t, x)) + DW_m(t, x) \cdot b(t, x, u, v) \\ + f_m(t, x, W_m(t, x), DW_m(t, x) \cdot \sigma(t, x, u, v), u, v) \} = 0, & (t, x) \in [0, T] \times \mathbb{R}^n, \\ W(T, x) = \Phi(x), & x \in \mathbb{R}^n, \end{cases} \quad (4.5)$$

where

$$f_m(t, x, y, z, u, v) = f(t, x, y, z, u, v) + m(y - h(t, x))^- , \\ (t, x, y, z, u, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times U \times V.$$

We have the uniqueness of viscosity solution W_m in the space $\tilde{\Theta}$ which is defined by

$$\tilde{\Theta} = \{ \varphi \in C([0, T] \times \mathbb{R}^n) : \exists \tilde{A} > 0 \text{ such that} \\ \lim_{|x| \rightarrow \infty} \varphi(t, x) \exp\{-\tilde{A}[\log((|x|^2 + 1)^{\frac{1}{2}})]^2\} = 0, \text{ uniformly in } t \in [0, T] \}.$$

Lemma 4.1. *For all $(t, x) \in [0, T] \times \mathbb{R}^n$ and all $m \geq 1$,*

$$W_1(t, x) \leq \dots \leq W_m(t, x) \leq W_{m+1}(t, x) \leq \dots \leq W(t, x).$$

Proof. Let $m \geq 1$. Since $f_m(t, x, y, z, u, v) \leq f_{m+1}(t, x, y, z, u, v)$, for all (t, x, y, z, u, v) we obtain from the comparison theorem for BSDEs (Lemma 2.2) that

$$J_m(t, x, u, v) = {}^m Y_t^{t,x;u,v} \leq {}^{m+1} Y_t^{t,x;u,v} = J_{m+1}(t, x, u, v), \text{ P-a.s., for any } u \in \mathcal{U}_{t,T} \text{ and } v \in \mathcal{V}_{t,T}.$$

Consequently, $W_m(t, x) \leq W_{m+1}(t, x)$, for all $(t, x) \in [0, T] \times \mathbb{R}^n$, $m \geq 1$.

From the result of the Section 6 [pp.719-pp.723] in [7], we can get that for each $0 \leq t \leq T$, $x \in \mathbb{R}^n$, $u \in \mathcal{U}_{t,T}$ and $v \in \mathcal{V}_{t,T}$,

$$J_m(t, x; u, v) \leq J(t, x; u, v), \text{ P-a.s.} \quad (4.6)$$

It follows that $W_m(t, x) \leq W(t, x)$, for all $(t, x) \in [0, T] \times \mathbb{R}^n$, $m \geq 1$. □

Remark 4.2. *The above lemma allows to introduce the lower semicontinuous function \widetilde{W} as limit over the non-decreasing sequence of continuous functions W_m , $m \geq 1$. From*

$$W_1(t, x) \leq \widetilde{W}(t, x) (= \lim_{m \uparrow \infty} \uparrow W_m(t, x)) \leq W(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^n,$$

and the linear growth of W_1 and W we conclude that also \widetilde{W} has at most linear growth.

Our objective is to prove that \widetilde{W} and W coincide and equation (4.1) holds in viscosity sense. For this end we first prove the following proposition:

Proposition 4.1. *Under the assumptions (H3.1) and (H3.2) the function $\widetilde{W}(t, x)$ is a viscosity supersolution of Isaacs equations (4.1).*

Proof. Let $(t, x) \in [0, T] \times \mathbb{R}^n$ and let $\varphi \in C_{l,b}^3([0, T] \times \mathbb{R}^n)$ be such that $\widetilde{W} - \varphi > \widetilde{W}(t, x) - \varphi(t, x)$ everywhere on $([0, T] \times \mathbb{R}^n) - \{(t, x)\}$. Then, since \widetilde{W} is lower semicontinuous and $W_m(t, x) \uparrow \widetilde{W}(t, x)$, $0 \leq t \leq T$, $x \in \mathbb{R}^n$, there exists some sequence (t_m, x_m) , $m \geq 1$, such that, at least along a subsequence,

- i) $(t_m, x_m) \rightarrow (t, x)$, as $m \rightarrow +\infty$;
- ii) $W_m - \varphi \geq W_m(t_m, x_m) - \varphi(t_m, x_m)$ in a neighborhood of (t_m, x_m) , for all $m \geq 1$;
- iii) $W_m(t_m, x_m) \rightarrow \widetilde{W}(t, x)$, as $m \rightarrow +\infty$.

Consequently, because W_m is a viscosity solution and hence a supersolution of equation (4.5), we have, for all $m \geq 1$,

$$\begin{aligned} & \frac{\partial}{\partial t} \varphi(t_m, x_m) + \sup_{u \in U} \inf_{v \in V} \left\{ \frac{1}{2} \text{tr}(\sigma \sigma^*(t_m, x_m, u, v) D^2 \varphi(t_m, x_m)) \right. \\ & + b(t_m, x_m, u, v) D \varphi(t_m, x_m) + f(t_m, x_m, W_m(t_m, x_m), D \varphi(t_m, x_m) \sigma(t_m, x_m, u, v), u, v) \} \\ & + m(W_m(t_m, x_m) - h(t_m, x_m))^- \end{aligned} \quad (4.7)$$

$$\leq 0.$$

Therefore,

$$\begin{aligned} & \frac{\partial}{\partial t} \varphi(t_m, x_m) + \sup_{u \in U} \inf_{v \in V} \left\{ \frac{1}{2} \text{tr}(\sigma \sigma^*(t_m, x_m, u, v) D^2 \varphi(t_m, x_m)) \right. \\ & + b(t_m, x_m, u, v) D \varphi(t_m, x_m) + f(t_m, x_m, W_m(t_m, x_m), D \varphi(t_m, x_m) \sigma(t_m, x_m, u, v), u, v) \} \\ & \leq 0. \end{aligned}$$

From $(t_m, x_m) \rightarrow (t, x)$ and $W_m(t_m, x_m) \rightarrow \widetilde{W}(t, x)$, as $m \rightarrow +\infty$, and the continuity of the functions b, σ and f and, hence, their uniform continuity on compacts (recall that U, V are compacts) it follows that, for all $m \geq 1$,

$$\begin{aligned} & \frac{\partial}{\partial t} \varphi(t_m, x_m) + \frac{1}{2} \text{tr}(\sigma \sigma^*(t_m, x_m, u, v) D^2 \varphi(t_m, x_m)) \\ & + b(t_m, x_m, u, v) D \varphi(t_m, x_m) + f(t_m, x_m, W_m(t_m, x_m), D \varphi(t_m, x_m) \sigma(t_m, x_m, u, v), u, v) \end{aligned}$$

converges uniformly in (u, v) towards

$$\begin{aligned} & \frac{\partial}{\partial t} \varphi(t, x) + \frac{1}{2} \text{tr}(\sigma \sigma^*(t, x, u, v) D^2 \varphi(t, x)) \\ & + b(t, x, u, v) D \varphi(t, x) + f(t, x, \widetilde{W}(t, x), D \varphi(t, x) \sigma(t, x, u, v), u, v). \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{\partial}{\partial t} \varphi(t, x) + \sup_{u \in U} \inf_{v \in V} \left\{ \frac{1}{2} \text{tr}(\sigma \sigma^*(t, x, u, v) D^2 \varphi(t, x)) \right. \\ & + b(t, x, u, v) D \varphi(t, x) + f(t, x, \widetilde{W}(t, x), D \varphi(t, x) \sigma(t, x, u, v), u, v) \} \\ & \leq 0. \end{aligned} \quad (4.8)$$

The above calculation shows that if $\widetilde{W}(t, x) \geq h(t, x)$ then we can conclude \widetilde{W} is a viscosity supersolution of (4.1). For this we return to the above inequality (4.7), from where

$$\begin{aligned} & -m(W_m(t_m, x_m) - h(t_m, x_m))^- \\ & \geq \frac{\partial}{\partial t} \varphi(t_m, x_m) + \sup_{u \in U} \inf_{v \in V} \left\{ \frac{1}{2} \text{tr}(\sigma \sigma^*(t_m, x_m, u, v) D^2 \varphi(t_m, x_m)) \right. \\ & + b(t_m, x_m, u, v) D \varphi(t_m, x_m) + f(t_m, x_m, W_m(t_m, x_m), D \varphi(t_m, x_m) \sigma(t_m, x_m, u, v), u, v) \}. \end{aligned}$$

When m tends to $+\infty$ the limit of the right-hand side of the above inequality, given by the left hand side of (4.8), is a real number. Therefore, the left-hand side of the above inequality cannot

tend to $-\infty$. But this is only possible if $(W_m(t_m, x_m) - h(t_m, x_m))^- \rightarrow 0$, i.e., if $\widetilde{W}(t, x) \geq h(t, x)$. The proof is complete. \square

Proposition 4.2. *Under the assumptions (H3.1) and (H3.2) the function $W(t, x)$ is a viscosity subsolution of Isaacs equations (4.1).*

Proof. We suppose that $\varphi \in C_{l,b}^3([0, T] \times \mathbb{R}^n)$ and $(t, x) \in [0, T] \times \mathbb{R}^n$ are such that $W - \varphi$ attains its maximum at (t, x) . Moreover, we assume that $\varphi(t, x) = W(t, x)$ and $W(t, x) > h(t, x)$. If the latter condition didn't hold we would have $W(t, x) = h(t, x)$ and thus also (4.1'). From the continuity of W and of h we conclude that there are some $r_0 > 0, \rho > 0$ such that $W(s, x') - h(s, x') \geq \rho$ for all $(s, x') \in [0, T] \times \mathbb{R}^n$ with $|(s, x') - (t, x)| \leq 2r_0$. On the other hand, by changing the test function φ outside the r_0 -neighborhood of (t, x) we can assume without loss of generality that $W(s, x') - \varphi(s, x') \leq -\rho$ for all $(s, x') \in [0, T] \times \mathbb{R}^n$ with $|(s, x') - (t, x)| \geq 2r_0$. Consequently, taking into account that $W \geq h$, we have everywhere on $[0, T] \times \mathbb{R}^n$ the relation $\varphi - h \geq \rho(> 0)$.

For getting (4.1') we shall prove that

$$F_0(t, x, 0, 0) := \sup_{u \in U} \inf_{v \in V} F(t, x, 0, 0, u, v) \geq 0,$$

where

$$F(s, x, y, z, u, v) = \frac{\partial}{\partial s} \varphi(s, x) + \frac{1}{2} \text{tr}(\sigma \sigma^T(s, x, u, v) D^2 \varphi) + D\varphi \cdot b(s, x, u, v) + f(s, x, y + \varphi(s, x), z + D\varphi(s, x) \cdot \sigma(s, x, u, v), u, v), \quad (4.9)$$

$$(s, x, y, z, u, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times U \times V.$$

Let us suppose that this is not true. Then there exists some $\theta > 0$ such that

$$F_0(t, x, 0, 0) = \sup_{u \in U} \inf_{v \in V} F(t, x, 0, 0, u, v) \leq -\theta < 0, \quad (4.10)$$

and we can find a measurable function $\psi : U \rightarrow V$ such that

$$F(t, x, 0, 0, u, \psi(u)) \leq -\theta, \text{ for all } u \in U.$$

Moreover, since $F(\cdot, x, 0, 0, \cdot, \cdot)$ is uniformly continuous on $[0, T] \times U \times V$ there exists some $T - t \geq R > 0$ such that

$$F(s, x, 0, 0, u, \psi(u)) \leq -\frac{1}{2}\theta, \text{ for all } u \in U \text{ and } |s - t| \leq R. \quad (4.11)$$

To continue the proof of Proposition 4.2 we need some auxiliary RBSDEs that are introduced and studied in the following:

Lemma 4.2. *For every $u \in \mathcal{U}_{t,t+\delta}$, $v \in \mathcal{V}_{t,t+\delta}$ and $s \in [t, t + \delta]$, we have the following relationship:*

$$Y_s^{1,u,v} = G_{s,t+\delta}^{t,x;u,v} [\varphi(t + \delta, X_{t+\delta}^{t,x;u,v})] - \varphi(s, X_s^{t,x;u,v}), \quad P\text{-a.s.}, \quad (4.12)$$

where $Y_s^{1,u,v}$ is the first component of the solution of the following RBSDE defined on the interval $[t, t + \delta]$ ($0 < \delta \leq T - t$):

$$\begin{cases} -dY_s^{1,u,v} = F(s, X_s^{t,x;u,v}, Y_s^{1,u,v}, Z_s^{1,u,v}, u_s, v_s) ds + dK_s^{1,u,v} - Z_s^{1,u,v} dB_s, \\ Y_{t+\delta}^{1,u,v} = 0, \quad Y_s^{1,u,v} \geq S_s := h(s, X_s^{t,x;u,v}) - \varphi(s, X_s^{t,x;u,v}), \quad a.s., \\ K_t^{1,u,v} = 0, \quad \int_t^{t+\delta} (Y_s^{1,u,v} - S_s) dK_s^{1,u,v} = 0, \end{cases} \quad (4.13)$$

Recall the process $X^{t,x,u,v}$ has been introduced by equation (3.1).

Remark 4.3. It's not hard to check that $F(s, X_s^{t,x;u,v}, y, z, u_s, v_s)$ satisfies (A1) and (A2). Thus, due to Lemma 2.3 equation (4.13) has a unique solution.

Proof. We recall that $G_{s,t+\delta}^{t,x;u,v}[\varphi(t+\delta, X_{t+\delta}^{t,x;u,v})]$ is defined with the help of the solution of the RBSDE

$$\left\{ \begin{array}{l} -d\tilde{Y}_s^{t,x;u,v} = f(s, X_s^{t,x;u,v}, \tilde{Y}_s^{t,x;u,v}, \tilde{Z}_s^{t,x;u,v}, u_s, v_s)ds + d\tilde{K}_s^{t,x;u,v} - \tilde{Z}_s^{t,x;u,v}dB_s, \\ \tilde{Y}_{t+\delta}^{t,x;u,v} = \varphi(t+\delta, X_{t+\delta}^{t,x;u,v}), \quad \tilde{Y}_s^{t,x;u,v} \geq h(s, X_s^{t,x;u,v}), \text{ a.s.}, \\ \tilde{K}_t^{t,x;u,v} = 0, \quad \int_t^{t+\delta} (\tilde{Y}_r^{t,x;u,v} - h(r, X_r^{t,x;u,v}))d\tilde{K}_r^{t,x;u,v} = 0, \end{array} \right.$$

by the following formula:

$$G_{s,t+\delta}^{t,x;u,v}[\varphi(t+\delta, X_{t+\delta}^{t,x;u,v})] = \tilde{Y}_s^{t,x;u,v}, \quad s \in [t, t+\delta] \quad (4.14)$$

(see (3.12)). Therefore, we only need to prove that $\tilde{Y}_s^{t,x;u,v} - \varphi(s, X_s^{t,x;u,v}) \equiv Y_s^{1,u,v}$. This result can be obtained easily by applying Itô's formula to $\varphi(s, X_s^{t,x;u,v})$. Indeed, we get that the stochastic differentials of $\tilde{Y}_s^{t,x;u,v} - \varphi(s, X_s^{t,x;u,v})$ and $Y_s^{1,u,v}$ coincide, while at the terminal time $t+\delta$, $\tilde{Y}_{t+\delta}^{t,x;u,v} - \varphi(t+\delta, X_{t+\delta}^{t,x;u,v}) = 0 = Y_{t+\delta}^{1,u,v}$, and $\tilde{Y}_s^{t,x;u,v} - \varphi(s, X_s^{t,x;u,v}) \geq h(s, X_s^{t,x;u,v}) - \varphi(s, X_s^{t,x;u,v}) = S_s$. Then from the uniqueness of the solution of the RBSDE the wished result follows. \square

Let us now consider the following RBSDE defined on the interval $[t, t+\delta]$ ($0 < \delta \leq T-t$) :

$$\left\{ \begin{array}{l} -dY_s^{2,u,v} = F(s, X_s^{t,x;u,v}, Y_s^{2,u,v}, Z_s^{2,u,v}, u_s, v_s)ds + dK_s^{2,u,v} - Z_s^{2,u,v}dB_s, \\ Y_{t+\delta}^{2,u,v} = 0, \quad Y_s^{2,u,v} \geq -\rho, \text{ a.s.}, \\ K_t^{2,u,v} = 0, \quad \int_t^{t+\delta} (Y_s^{2,u,v} + \rho)dK_s^{2,u,v} = 0, \end{array} \right. \quad (4.15)$$

where $u(\cdot) \in \mathcal{U}_{t,t+\delta}$, $v(\cdot) \in \mathcal{V}_{t,t+\delta}$.

Lemma 4.3. For every $s \in [t, t+\delta]$, $Y_s^{1,u,v} \leq Y_s^{2,u,v}$, a.s..

Proof. Notice that $h(s, y) - \varphi(s, y) \leq -\rho$, for all $(s, y) \in [t, T] \times \mathbb{R}^n$. Therefore, the above assertion follows directly from Lemma 2.4 -the comparison theorem for RBSDEs. \square

Finally we still study the following simpler RBSDE in which the driving process $X^{t,x;u,v}$ is replaced by its deterministic initial value x :

$$\left\{ \begin{array}{l} -dY_s^{3,u,v} = F(s, x, Y_s^{3,u,v}, Z_s^{3,u,v}, u_s, v_s)ds + dK_s^{3,u,v} - Z_s^{3,u,v}dB_s, \\ Y_{t+\delta}^{3,u,v} = 0, \quad Y_s^{3,u,v} \geq -\rho, \text{ a.s.}, \\ K_t^{3,u,v} = 0, \quad \int_t^{t+\delta} (Y_s^{3,u,v} + \rho)dK_s^{3,u,v} = 0, \end{array} \right. \quad (4.16)$$

where $u(\cdot) \in \mathcal{U}_{t,t+\delta}$, $v(\cdot) \in \mathcal{V}_{t,t+\delta}$. The following lemma will allow us to neglect the difference $|Y_t^{2,u,v} - Y_t^{3,u,v}|$ for sufficiently small $\delta > 0$.

Lemma 4.4. For every $u \in \mathcal{U}_{t,t+\delta}$, $v \in \mathcal{V}_{t,t+\delta}$, we have

$$|Y_t^{2,u,v} - Y_t^{3,u,v}| \leq C\delta^{\frac{3}{2}}, \quad P\text{-a.s.}, \quad (4.17)$$

where C is independent of the control processes u and v .

Proof. From (3.3) we have for all $p \geq 2$ the existence of some $C_p \in \mathbb{R}_+$ such that

$$E\left[\sup_{t \leq s \leq T} |X_s^{t,x;u,v}|^p | \mathcal{F}_t\right] \leq C_p(1 + |x|^p), \quad \text{P-a.s., uniformly in } u \in \mathcal{U}_{t,t+\delta}, v \in \mathcal{V}_{t,t+\delta}.$$

This combined with the estimate

$$\begin{aligned} E\left[\sup_{t \leq s \leq t+\delta} |X_s^{t,x;u,v} - x|^p | \mathcal{F}_t\right] &\leq 2^{p-1} E\left[\sup_{t \leq s \leq t+\delta} \left|\int_t^s b(r, X_r^{t,x;u,v}, u_r, v_r) dr\right|^p | \mathcal{F}_t\right] \\ &\quad + 2^{p-1} E\left[\sup_{t \leq s \leq t+\delta} \left|\int_t^s \sigma(r, X_r^{t,x;u,v}, u_r, v_r) dB_r\right|^p | \mathcal{F}_t\right] \end{aligned}$$

yields

$$E\left[\sup_{t \leq s \leq t+\delta} |X_s^{t,x;u,v} - x|^p | \mathcal{F}_t\right] \leq C_p \delta^{\frac{p}{2}}, \quad \text{P-a.s., uniformly in } u \in \mathcal{U}_{t,t+\delta}, v \in \mathcal{V}_{t,t+\delta}. \quad (4.18)$$

We now apply Lemma 2.6 combined with (4.18) to equations (4.15) and (4.16). For this we set in Lemma 2.6:

$$\xi_1 = \xi_2 = 0, \quad g_1(s, y, z) = F(s, X_s^{t,x,u,v}, y, z, u_s, v_s), \quad g_2(s, y, z) = F(s, x, y, z, u_s, v_s),$$

$$S_1 = S_2 = -\rho, \quad \Delta g(s, Y_s^{2,u,v}, Z_s^{2,u,v}) = g_1(s, Y_s^{2,u,v}, Z_s^{2,u,v}) - g_2(s, Y_s^{2,u,v}, Z_s^{2,u,v}).$$

Obviously, the functions g_1 and g_2 are Lipschitz with respect to (y, z) , and $|\Delta g(s, Y_s^{2,u,v}, Z_s^{2,u,v})| \leq C(1 + |x|^2)(|X_s^{t,x;u,v} - x| + |X_s^{t,x;u,v} - x|^3)$, for $s \in [t, t+\delta]$, $(t, x) \in [0, T] \times \mathbb{R}^n$, $u \in \mathcal{U}_{t,t+\delta}$, $v \in \mathcal{V}_{t,t+\delta}$. Thus, with the notation $\rho_0(r) = (1 + |x|^2)(r + r^3)$, $r \geq 0$, we have

$$\begin{aligned} |Y_t^{2,u,v} - Y_t^{3,u,v}|^2 &= |E[|Y_t^{2,u,v} - Y_t^{3,u,v}|^2 | \mathcal{F}_t]| \\ &\leq CE\left[\left(\int_t^{t+\delta} |\Delta g(s, Y_s^{2,u,v}, Z_s^{2,u,v})| ds\right)^2 | \mathcal{F}_t\right] \\ &\leq C\delta E\left[\int_t^{t+\delta} |\Delta g(s, Y_s^{2,u,v}, Z_s^{2,u,v})|^2 ds | \mathcal{F}_t\right] \\ &\leq C\delta E\left[\int_t^{t+\delta} \rho_0^2(|X_s^{t,x,u,v} - x|) ds | \mathcal{F}_t\right] \\ &\leq C\delta^2 E\left[\sup_{t \leq s \leq t+\delta} \rho_0^2(|X_s^{t,x,u,v} - x|) | \mathcal{F}_t\right] \\ &\leq C\delta^3. \end{aligned}$$

Thus, the proof is complete. \square

Lemma 4.5. *There is some $\delta_0 > 0$ such that, for all $\delta \in (0, \delta_0]$ and for every $u \in \mathcal{U}_{t,t+\delta}$, we have*

$$Y_t^{3,u,\psi(u)} \leq -\frac{\theta}{2C} \left(1 - e^{-C\delta}\right), \quad \text{P-a.s.} \quad (4.19)$$

$C > 0$ is the Lipschitz constant of F and thus in particular independent of the controls u and also of δ . Here, by putting $\psi_s(u)(\omega) = \psi(u_s(\omega))$, $(s, \omega) \in [t, T] \times \Omega$, we identify ψ as an element of $\mathcal{B}_{t,t+\delta}$.

Proof. We observe that, if $\delta \leq R$, for all $(s, y, z, u) \in [t, t+\delta] \times \mathbb{R} \times \mathbb{R}^d \times U$, from (4.11)

$$\begin{aligned} F(s, x, y, z, u, \psi(u)) &\leq C(|y| + |z|) + F(s, x, 0, 0, u, \psi(u)) \\ &\leq C(|y| + |z|) - \frac{1}{2}\theta. \end{aligned}$$

Consequently, from the comparison result for RBSDEs (Lemma 2.4) we have that $Y_s^{3,u,\psi(u)} \leq Y_s^4$, $s \in [t, t + \delta]$, where Y^4 is defined by the following RBSDE:

$$\begin{cases} -dY_s^4 = \{C(|Y_s^4| + |Z_s^4|) - \frac{1}{2}\theta\}ds + dK_s^4 - Z_s^4dB_s, \\ Y_{t+\delta}^4 = 0, \quad Y_s^4 \geq -\rho, \text{ a.s.}, \\ K_t^4 = 0, \quad \int_t^{t+\delta} (Y_s^4 + \rho)dK_s^4 = 0. \end{cases} \quad (4.20)$$

But, obviously, for $\delta \in (0, \delta_0]$ with $\delta_0 > 0$ small enough such that $\frac{\theta}{2C} (1 - e^{-C\delta_0}) < \rho$, the unique solution of this RBSDE is given by

$$Y_s^4 = -\frac{\theta}{2C} \left(1 - e^{C(s-(t+\delta))}\right), \quad Z_s^4 = 0, \quad K_s^4 = 0, \quad s \in [t, t + \delta].$$

The assertion of the lemma follows now easily. \square

The above auxiliary results now allow to complete the proof of Proposition 4.2.

Proof of Proposition 4.2 (sequel). Due to the DPP (see Theorem 3.1), for every $\delta \in (0, \delta_0]$,

$$\varphi(t, x) = W(t, x) = \text{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \text{esssup}_{u \in \mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{t,x;u,\beta(u)}[W(t + \delta, X_{t+\delta}^{t,x;u,\beta(u)})],$$

and from $W(s, y) \leq \varphi(s, y)$, for all $(s, y) \in [0, T] \times \mathbb{R}^n$, and the monotonicity property of $G_{t,t+\delta}^{t,x;u,\beta(u)}[\cdot]$ (see Lemma 2.4) we obtain

$$\text{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \text{esssup}_{u \in \mathcal{U}_{t,t+\delta}} \{G_{t,t+\delta}^{t,x;u,\beta(u)}[\varphi(t + \delta, X_{t+\delta}^{t,x;u,\beta(u)})] - \varphi(t, x)\} \geq 0, \text{ P-a.s.}$$

Thus, from Lemma 4.2,

$$\text{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \text{esssup}_{u \in \mathcal{U}_{t,t+\delta}} Y_t^{1,u,\beta(u)} \geq 0, \text{ P-a.s.},$$

and from Lemma 4.3,

$$\text{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \text{esssup}_{u \in \mathcal{U}_{t,t+\delta}} Y_t^{2,u,\beta(u)} \geq 0, \text{ P-a.s.}$$

Then, in particular,

$$\text{esssup}_{u \in \mathcal{U}_{t,t+\delta}} Y_t^{2,u,\psi(u)} \geq 0, \text{ P-a.s.}$$

Hence, given an arbitrary $\varepsilon > 0$ we can choose $u^\varepsilon \in \mathcal{U}_{t,t+\delta}$ such that, P-a.s.,

$$Y_t^{2,u^\varepsilon,\psi(u^\varepsilon)} \geq -\varepsilon\delta.$$

(The proof is similar to that of inequality (7.3) in the Appendix II.) Consequently, from Lemma 4.4,

$$Y_t^{3,u^\varepsilon,\psi(u^\varepsilon)} \geq -C\delta^{\frac{3}{2}} - \varepsilon\delta, \text{ P-a.s.} \quad (4.21)$$

By combining this result with Lemma 4.5 we then obtain

$$-C\delta^{\frac{3}{2}} - \varepsilon\delta \leq Y_t^{3,u^\varepsilon,\psi(u^\varepsilon)} \leq -\frac{\theta}{2C} (1 - e^{-C\delta}), \text{ P-a.s.}$$

Therefore,

$$-C\delta^{\frac{1}{2}} - \varepsilon \leq -\frac{\theta}{2C} \frac{1 - e^{-C\delta}}{\delta},$$

and by taking the limit as $\delta \downarrow 0, \varepsilon \downarrow 0$ we get $0 \leq -\frac{\theta}{2}$ which contradicts our assumption that $\theta > 0$. Therefore, it must hold that

$$F_0(t, x, 0, 0) = \sup_{u \in U} \inf_{v \in V} F(t, x, 0, 0, u, v) \geq 0,$$

and from the definition of F it follows that W is a viscosity subsolution of equation (4.1). \square

Proof of Theorem 4.1. From Theorem 5.1 which is proved in Section 5, Propositions 4.1 and 4.2 we get $W(t, x) \leq \widehat{W}(t, x)$. Furthermore, from Lemma 4.1 we obtain $W(t, x) = \widehat{W}(t, x)$. The proof is complete. \square

As a byproduct of the proof of Theorem 5.1 we have that the viscosity solution W_m of Isaacs equation (4.5) converges pointwise to the viscosity solution of Isaacs equation with obstacles (4.1):

Theorem 4.2. $W_m(t, x) \uparrow W(t, x)$, as $m \rightarrow +\infty$, for any $(t, x) \in [0, T] \times \mathbb{R}^n$.

5 Viscosity Solution of Isaacs' Equation with obstacle: Uniqueness Theorem

The objective of this section is to study the uniqueness of the viscosity solution of Isaacs' equation (4.1),

$$\begin{cases} \min\{W(t, x) - h(t, x), -\frac{\partial}{\partial t}W(t, x) - H^-(t, x, W, DW, D^2W)\} = 0, & (t, x) \in [0, T] \times \mathbb{R}^n, \\ W(T, x) = \Phi(x), & x \in \mathbb{R}^n, \end{cases} \quad (5.1)$$

associated with the Hamiltonians

$$H^-(t, x, y, q, X) = \sup_{u \in U} \inf_{v \in V} \left\{ \frac{1}{2} \text{tr}(\sigma \sigma^T(t, x, u, v)X) + q \cdot b(t, x, u, v) + f(t, x, y, q, \sigma, u, v) \right\},$$

$t \in [0, T]$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $q \in \mathbb{R}^n$, $X \in \mathbb{S}^n$. The functions b, σ, f and Φ are still supposed to satisfy (H3.1) and (H3.2), respectively.

For the proof of the uniqueness of the viscosity solution we borrow the main idea from Barles, Buckdahn, Pardoux [1]. Similarly, we will prove the uniqueness for equation (5.1) in the space of functions

$$\Theta = \{\varphi : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R} \mid \exists \tilde{A} > 0 \text{ such that} \\ \lim_{|x| \rightarrow \infty} \varphi(t, x) \exp\{-\tilde{A}[\log((|x|^2 + 1)^{\frac{1}{2}})]^2\} = 0, \text{ uniformly in } t \in [0, T]\}.$$

This growth condition was introduced in [1] to prove the uniqueness of the viscosity solution of an integro-partial differential equation associated with a decoupled FBSDE with jumps. It was shown in [1] that this kind of growth condition is optimal for the uniqueness and can, in general, not be weakened, even not for PDEs. We adapt the ideas developed in [1] to Isaacs' equation (5.1) to prove the uniqueness of the viscosity solution in Θ . Since the proof of the uniqueness in Θ for equation (4.2) is the same we will restrict ourselves only on that of (5.1). Before stating the main result of this section, let us begin with two auxiliary lemmata. Denoting by K a Lipschitz constant of $f(t, x, \cdot, \cdot)$, that is uniformly in (t, x) , we have the following

Lemma 5.1. *Let an upper semicontinuous function $u_1 \in \Theta$ be a viscosity subsolution and a lower semicontinuous function $u_2 \in \Theta$ be a viscosity supersolution of equation (5.1). Then, the upper*

semicontinuous function $\omega := u_1 - u_2$ is a viscosity subsolution of the equation

$$\begin{cases} \min\{\omega(t, x), -\frac{\partial}{\partial t}\omega(t, x) - \sup_{u \in U, v \in V}(\frac{1}{2}\text{tr}(\sigma\sigma^T(t, x, u, v)D^2\omega) + D\omega \cdot b(t, x, u, v) + K|\omega| \\ \quad + K|D\omega \cdot \sigma(t, x, u, v)|)\} = 0, & (t, x) \in [0, T] \times \mathbb{R}^n, \\ \omega(T, x) = 0, & x \in \mathbb{R}^n. \end{cases} \quad (5.2)$$

Proof. The proof is similar to that of Lemma 3.7 in [1], the main difference consists in the fact that here we have to deal with an obstacle problem and u_1, u_2 are not continuous. First we notice that $\omega(T, x) = u_1(T, x) - u_2(T, x) \leq \Phi(x) - \Phi(x) = 0$. For $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$ let $\varphi \in C_{l,b}^3([0, T] \times \mathbb{R}^n)$ be such that (t_0, x_0) be a strict global maximum point of $w - \varphi$. Separating the variables we introduce the function

$$\Phi_{\varepsilon, \alpha}(t, x, s, y) = u_1(t, x) - u_2(s, y) - \frac{|x - y|^2}{\varepsilon^2} - \frac{(t - s)^2}{\alpha^2} - \varphi(t, x),$$

where ε, α are positive parameters which are devoted to tend to zero.

Since (t_0, x_0) is a strict global maximum point of $w - \varphi$, there exists a sequence $(\bar{t}, \bar{x}, \bar{s}, \bar{y})$ such that

(i) $(\bar{t}, \bar{x}, \bar{s}, \bar{y})$ is a global maximum point of $\Phi_{\varepsilon, \alpha}$ in $[0, T] \times \bar{B}_r \times \bar{B}_r$ where B_r is a ball with a large radius r ;

(ii) $(\bar{t}, \bar{x}), (\bar{s}, \bar{y}) \rightarrow (t_0, x_0)$ as $(\varepsilon, \alpha) \rightarrow 0$;

(iii) $\frac{|\bar{x} - \bar{y}|^2}{\varepsilon^2}, \frac{(\bar{t} - \bar{s})^2}{\alpha^2}$ are bounded and tend to zero when $(\varepsilon, \alpha) \rightarrow 0$.

Since u_2 is lower semicontinuous we have $\lim_{(\varepsilon, \alpha) \rightarrow 0} u_2(\bar{s}, \bar{y}) \geq u_2(t_0, x_0)$, and u_1 is upper semicontinuous we have $\lim_{(\varepsilon, \alpha) \rightarrow 0} u_1(\bar{t}, \bar{x}) \leq u_1(t_0, x_0)$. On the other hand, from $\Phi_{\varepsilon, \alpha}(\bar{t}, \bar{x}, \bar{s}, \bar{y}) \geq \Phi_{\varepsilon, \alpha}(t_0, x_0, t_0, x_0)$ we get

$$u_2(\bar{s}, \bar{y}) \leq u_1(\bar{t}, \bar{x}) - u_1(t_0, x_0) + u_2(t_0, x_0) + \varphi(t_0, x_0) - \varphi(\bar{t}, \bar{x}) - \frac{|\bar{x} - \bar{y}|^2}{\varepsilon^2} - \frac{(\bar{t} - \bar{s})^2}{\alpha^2},$$

and from where we have $\lim_{(\varepsilon, \alpha) \rightarrow 0} u_2(\bar{s}, \bar{y}) \leq u_2(t_0, x_0)$. Therefore, we have

(iv) $\lim_{(\varepsilon, \alpha) \rightarrow 0} u_2(\bar{s}, \bar{y}) = u_2(t_0, x_0)$.

Analogously, we also get

(v) $\lim_{(\varepsilon, \alpha) \rightarrow 0} u_1(\bar{t}, \bar{x}) = u_1(t_0, x_0)$.

Since $(\bar{t}, \bar{x}, \bar{s}, \bar{y})$ is a local maximum point of $\Phi_{\varepsilon, \alpha}$, $u_2(s, y) + \frac{|\bar{x} - y|^2}{\varepsilon^2} + \frac{(\bar{t} - s)^2}{\alpha^2}$ achieves in (\bar{s}, \bar{y}) a local minimum and from the definition of a viscosity supersolution of equation (4.1) we have $u_2(\bar{s}, \bar{y}) \geq h(\bar{s}, \bar{y})$. From (iv) we get $u_2(t_0, x_0) \geq h(t_0, x_0)$. Hence, if $u_1(t_0, x_0) \leq h(t_0, x_0)$ we have

$$w(t_0, x_0) = u_1(t_0, x_0) - u_2(t_0, x_0) \leq 0,$$

and the proof is complete. Therefore, in the sequel, we only need to consider the case that $u_1(t_0, x_0) > h(t_0, x_0)$. Then, according to (v) and because h is continuous we can require

(vi) $u_1(\bar{t}, \bar{x}) > h(\bar{t}, \bar{x})$, for $\varepsilon > 0$ and $\alpha > 0$ sufficiently small.

The properties (i) to (vi) and the fact that u_1 is a viscosity subsolution and u_2 a viscosity supersolution of equation (5.1) allow to proceed in the rest of the proof of this lemma exactly as in

the proof of Lemma 3.7 in [1] (our situation here is even simpler because contrary to Lemma 3.7 in [1], we don't have any integral part in equation (5.1)). So we get:

$$-\frac{\partial \varphi}{\partial t}(t_0, x_0) - \sup_{u \in U, v \in V} \left\{ \frac{1}{2} \text{tr} \left((\sigma \sigma^T)(t_0, x_0, u, v) D^2 \varphi(t_0, x_0) \right) + D\varphi(t_0, x_0) b(t_0, x_0, u, v) \right. \\ \left. + K|\omega(t_0, x_0)| + K|D\varphi(t_0, x_0) \sigma(t_0, x_0, u, v)| \right\} \leq 0.$$

Therefore ω is a viscosity subsolution of the desired equation (5.2) and the proof is complete. \square

Theorem 5.1. *We assume that (H3.1) and (H3.2) hold. Let an upper semicontinuous function u_1 (resp., a lower semicontinuous function u_2) $\in \Theta$ be a viscosity subsolution (resp., supersolution) of equation (5.1). Then we have*

$$u_1(t, x) \leq u_2(t, x), \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}^n. \quad (5.3)$$

Proof. Let us put $\omega := u_1 - u_2$. Then we have, from Lemma 5.1 ω is a viscosity subsolution of equation (5.2). On the other hand, $\omega' = 0$ is a viscosity solution of (5.2). Then, from the comparison principle for Hamilton-Jacobi-Bellman equations with standard assumptions on the coefficients (see, for instance, [19]) it follows that $\omega \leq \omega' = 0$. Thus, the proof is complete. \square

Remark 5.1. *Obviously, since the lower value function $W(t, x)$ and $\tilde{W}(t, x) = \lim_{m \rightarrow \infty} \uparrow W_m(t, x) (\leq W(t, x))$ (for the definition, see Lemma 4.1), are a viscosity subsolution and a supersolution, respectively (see Proposition 4.1 and 4.2), both are of linear growth and $\tilde{W} \leq W$, we have from Theorem 5.1 that $W(t, x) = \tilde{W}(t, x)$, $(t, x) \in [0, T] \times \mathbb{R}^n$. Similarly we get that the upper value function $U(t, x)$ is the unique viscosity solution in Θ of equation (4.2). On the other hand, since $H^- \leq H^+$, any viscosity solution of equation (4.2) is a supersolution of equation (5.1). Then, again from Theorem 5.1, it follows that $W \leq U$. This justifies calling W lower value function and U upper value function.*

Remark 5.2. *If the Isaacs' condition holds, that is, if for all $(t, x, y, p, X) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$,*

$$H^-(t, x, y, p, X) = H^+(t, x, y, p, X),$$

then the equations (5.1) and (4.2) coincide and from the uniqueness of the viscosity solution in Θ it follows that the lower value function $W(t, x)$ equals to the upper value function $U(t, x)$, that means the associated stochastic differential game with reflections has a value.

6 Appendix I: RBSDEs Associated with Forward SDEs

In this section we give an overview over basic results on RBSDEs associated with Forward SDEs (for short: FSDEs). This overview includes also new results (Proposition 6.1) playing a crucial role in the approach developed in this paper.

We consider measurable functions $b : [0, T] \times \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma : [0, T] \times \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$ which are supposed to satisfy the following conditions:

- (i) $b(\cdot, 0)$ and $\sigma(\cdot, 0)$ are \mathbb{F} -adapted processes, and there exists some constant $C > 0$ such that
$$|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|), \text{ a.s., for all } 0 \leq t \leq T, x \in \mathbb{R}^n; \quad (\text{H6.1})$$
- (ii) b and σ are Lipschitz in x , i.e., there is some constant $C > 0$ such that
$$|b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq C|x - x'|, \text{ a.s.,}$$

for all $0 \leq t \leq T, x, x' \in \mathbb{R}^n$.

We now consider the following SDE parameterized by the initial condition $(t, \zeta) \in [0, T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$:

$$\begin{cases} dX_s^{t, \zeta} &= b(s, X_s^{t, \zeta})ds + \sigma(s, X_s^{t, \zeta})dB_s, \quad s \in [t, T], \\ X_t^{t, \zeta} &= \zeta. \end{cases} \quad (6.1)$$

Under the assumption (H6.1), SDE (6.1) has a unique strong solution and, for any $p \geq 2$, there exists $C_p \in \mathbb{R}$ such that, for any $t \in [0, T]$ and $\zeta, \zeta' \in L^p(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$,

$$\begin{aligned} E[\sup_{t \leq s \leq T} |X_s^{t, \zeta} - X_s^{t, \zeta'}|^p | \mathcal{F}_t] &\leq C_p |\zeta - \zeta'|^p, \quad a.s., \\ E[\sup_{t \leq s \leq T} |X_s^{t, \zeta}|^p | \mathcal{F}_t] &\leq C_p (1 + |\zeta|^p), \quad a.s. \end{aligned} \quad (6.2)$$

These well-known standard estimates can be consulted, for instance, in Ikeda, Watanabe [14], pp.166-168 and also in Karatzas, Shreve [15], pp.289-290. We emphasize that the constant C_p in (6.2) only depends on the Lipschitz and the growth constants of b and σ .

Let now be given three real valued functions $f(t, x, y, z)$, $\Phi(x)$ and $h(t, x)$ which shall satisfy the following conditions:

- (i) $\Phi : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ is an $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^n)$ -measurable random variable and $f : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, $h : \Omega \times [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ are measurable processes such that, $f(\cdot, x, y, z)$, $h(\cdot, x)$ are \mathcal{F}_t -adapted, for all $(x, y, z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$;
- (ii) There exists a constant $\mu > 0$ such that, P-a.s.,

$$\begin{aligned} |f(t, x, y, z) - f(t, x', y', z')| &\leq \mu(|x - x'| + |y - y'| + |z - z'|); \\ |\Phi(x) - \Phi(x')| &\leq \mu|y - y'|; \\ |h(t, x) - h(t, x')| &\leq \mu|x - x'|; \end{aligned}$$
 for all $0 \leq t \leq T$, $x, x' \in \mathbb{R}^n$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^d$;
- (iii) f and Φ satisfy a linear growth condition, i.e., there exists some $C > 0$ such that, $dt \times dP$ -a.e., for all $x \in \mathbb{R}^n$,

$$|f(t, x, 0, 0)| + |\Phi(x)| \leq C(1 + |x|) \quad (H6.2)$$
 and, moreover, $h(\cdot, x)$ is continuous in t and $h(T, x) \leq \Phi(x)$ a.s., for all $x \in \mathbb{R}^n$.

With the help of the above assumptions we can verify that the coefficient $f(s, X_s^{t, \zeta}, y, z)$ satisfies the hypotheses (A1), (A2), $\xi = \Phi(X_T^{t, \zeta}) \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})$ and $S_s = h(s, X_s^{t, \zeta})$ fulfills (A3). Therefore, the following RBSDE possesses a unique solution:

$$\begin{aligned} (i) & Y^{t, \zeta} \in \mathcal{S}^2(0, T; \mathbb{R}), \quad Z^{t, \zeta} \in \mathcal{H}^2(0, T; \mathbb{R}^d) \text{ and } K_T^{t, \zeta} \in L^2(\Omega, \mathcal{F}_T, P); \\ (ii) & Y_s^{t, \zeta} = \Phi(X_T^{t, \zeta}) + \int_s^T f(r, X_r^{t, \zeta}, Y_r^{t, \zeta}, Z_r^{t, \zeta})dr + K_T^{t, \zeta} - K_s^{t, \zeta} - \int_s^T Z_r^{t, \zeta} dB_r, \quad s \in [t, T]; \\ (iii) & Y_s^{t, \zeta} \geq h(s, X_s^{t, \zeta}), \quad a.s., \text{ for any } s \in [t, T]; \\ (iv) & K^{t, \zeta} \text{ is continuous and increasing, } K_t^{t, \zeta} = 0, \quad \int_t^T (Y_r^{t, \zeta} - h(r, X_r^{t, \zeta}))dK_r^{t, \zeta} = 0. \end{aligned} \quad (6.3)$$

Proposition 6.1. *We suppose that the hypotheses (H6.1) and (H6.2) hold. Then, for any $0 \leq t \leq T$ and the associated initial conditions $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we have the following estimates:*

$$(i) E[\sup_{t \leq s \leq T} |Y_s^{t, \zeta}|^2 + \int_t^T |Z_s^{t, \zeta}|^2 ds + |K_T^{t, \zeta}|^2 | \mathcal{F}_t] \leq C(1 + |\zeta|^2), \quad a.s.;$$

$$(ii) E[\sup_{t \leq s \leq T} |Y_s^{t,\zeta} - Y_s^{t,\zeta'}|^2 | \mathcal{F}_t] \leq C |\zeta - \zeta'|^2, \text{ a.s.}$$

In particular,

$$\begin{aligned} (iii) \quad & |Y_t^{t,\zeta}| \leq C(1 + |\zeta|), \text{ a.s.}; \\ (iv) \quad & |Y_t^{t,\zeta} - Y_t^{t,\zeta'}| \leq C|\zeta - \zeta'|, \text{ a.s.} \end{aligned} \tag{6.4}$$

The above constant $C > 0$ depends only on the Lipschitz and the growth constants of b, σ, f, Φ and h .

Proof. From Lemma 2.5 we get (i). So we need only to prove (ii). For an arbitrarily fixed $\varepsilon > 0$, we define the function $\psi_\varepsilon(x) = (|x|^2 + \varepsilon)^{\frac{1}{2}}$, $x \in \mathbb{R}^n$. Obviously, $|x| \leq \psi_\varepsilon(x) \leq |x| + \varepsilon^{\frac{1}{2}}$, $x \in \mathbb{R}^n$. Furthermore, for all $x \in \mathbb{R}^n$,

$$D\psi_\varepsilon(x) = \frac{x}{(|x|^2 + \varepsilon)^{\frac{1}{2}}}, \quad D^2\psi_\varepsilon(x) = \frac{I}{(|x|^2 + \varepsilon)^{\frac{1}{2}}} - \frac{x \otimes x}{(|x|^2 + \varepsilon)^{\frac{3}{2}}}.$$

Then, we have

$$|D\psi_\varepsilon(x)| \leq 1, \quad |D^2\psi_\varepsilon(x)||x| \leq \frac{C}{(|x|^2 + \varepsilon)^{\frac{1}{2}}}|x| \leq C, \quad x \in \mathbb{R}^n, \tag{6.5}$$

where the constant C is independent of ε . Let us denote by $X^{t,\zeta}$ and $X^{t,\zeta'}$ the unique solution of SDE (6.1) with the initial data (t, ζ) and (t, ζ') , respectively. Moreover, recall that μ is the Lipschitz constant of h, Φ , and f . We consider the following two RBSDEs:

$$\begin{aligned} (i) \quad & \tilde{Y} \in \mathcal{S}^2(0, T; \mathbb{R}), \quad \tilde{Z} \in \mathcal{H}^2(0, T; \mathbb{R}^d) \quad \text{and} \quad \tilde{K}_T \in L^2(\Omega, \mathcal{F}_T, P); \\ (ii) \quad & \tilde{Y}_s = \Phi(X_s^{t,\zeta}) + \mu\psi_\varepsilon(X_s^{t,\zeta} - X_s^{t,\zeta'}) + \int_s^T (f(r, X_r^{t,\zeta}, \tilde{Y}_r, \tilde{Z}_r) + \mu|X_r^{t,\zeta} - X_r^{t,\zeta'}|)dr \\ & \quad + \tilde{K}_T - \tilde{K}_s - \int_s^T \tilde{Z}_r dB_r, \quad s \in [t, T]; \\ (iii) \quad & \tilde{Y}_s \geq h(s, X_s^{t,\zeta}) + \mu\psi_\varepsilon(X_s^{t,\zeta} - X_s^{t,\zeta'}), \quad \text{a.s., for any } s \in [t, T]; \\ (iv) \quad & \tilde{K} \text{ is continuous and increasing, } \tilde{K}_t = 0, \\ & \quad \int_t^T (\tilde{Y}_r - h(r, X_r^{t,\zeta}) - \mu\psi_\varepsilon(X_r^{t,\zeta} - X_r^{t,\zeta'}))d\tilde{K}_r = 0. \end{aligned} \tag{6.6}$$

and

$$\begin{aligned} (i) \quad & \bar{Y} \in \mathcal{S}^2(0, T; \mathbb{R}), \quad \bar{Z} \in \mathcal{H}^2(0, T; \mathbb{R}^d) \quad \text{and} \quad \bar{K}_T \in L^2(\Omega, \mathcal{F}_T, P); \\ (ii) \quad & \bar{Y}_s = \Phi(X_s^{t,\zeta}) - \mu|X_s^{t,\zeta} - X_s^{t,\zeta'}| + \int_s^T (f(r, X_r^{t,\zeta}, \bar{Y}_r, \bar{Z}_r) - \mu|X_r^{t,\zeta} - X_r^{t,\zeta'}|)dr \\ & \quad + \bar{K}_T - \bar{K}_s - \int_s^T \bar{Z}_r dB_r, \quad s \in [t, T]; \\ (iii) \quad & \bar{Y}_s \geq h(s, X_s^{t,\zeta}) - \mu\psi_\varepsilon(X_s^{t,\zeta} - X_s^{t,\zeta'}), \quad \text{a.s., for any } s \in [t, T]; \\ (iv) \quad & \bar{K} \text{ is continuous and increasing, } \bar{K}_t = 0, \\ & \quad \int_t^T (\bar{Y}_r - h(r, X_r^{t,\zeta}) + \mu\psi_\varepsilon(X_r^{t,\zeta} - X_r^{t,\zeta'}))d\bar{K}_r = 0. \end{aligned} \tag{6.7}$$

Obviously, their coefficients satisfy the assumptions in (H6.2) and they admit unique solutions $(\tilde{Y}, \tilde{Z}, \tilde{K})$ and $(\bar{Y}, \bar{Z}, \bar{K})$, respectively. Moreover, from the comparison theorem for RBSDEs (Lemma 2.4)

$$\bar{Y}_s \leq Y_s^{t,\zeta} \leq \tilde{Y}_s, \quad \bar{Y}_s \leq Y_s^{t,\zeta'} \leq \tilde{Y}_s, \quad \text{P-a.s., for all } s \in [t, T]. \tag{6.8}$$

We shall introduce two other RBSDEs:

$$\begin{aligned}
& \text{(i)} \tilde{Y}' \in \mathcal{S}^2(0, T; \mathbb{R}), \tilde{Z}' \in \mathcal{H}^2(0, T; \mathbb{R}^d) \text{ and } \tilde{K}'_T \in L^2(\Omega, \mathcal{F}_T, P); \\
& \text{(ii)} \tilde{Y}'_s = \Phi(X_T^{t, \zeta}) + \\
& \quad \int_s^T [f(r, X_r^{t, \zeta}, \tilde{Y}'_r + \mu\psi_\varepsilon(X_r^{t, \zeta} - X_r^{t, \zeta'}), \tilde{Z}'_r + \mu D\psi_\varepsilon(X_r^{t, \zeta} - X_r^{t, \zeta'})(\sigma(r, X_r^{t, \zeta}) - \sigma(r, X_r^{t, \zeta'}))) \\
& \quad + \mu|X_r^{t, \zeta} - X_r^{t, \zeta'}| + \mu D\psi_\varepsilon(X_r^{t, \zeta} - X_r^{t, \zeta'})(b(r, X_r^{t, \zeta}) - b(r, X_r^{t, \zeta'})) \\
& \quad + \frac{1}{2}\mu(D^2\psi_\varepsilon(X_r^{t, \zeta} - X_r^{t, \zeta'})(\sigma(r, X_r^{t, \zeta}) - \sigma(r, X_r^{t, \zeta'})), \sigma(r, X_r^{t, \zeta}) - \sigma(r, X_r^{t, \zeta'}))]dr \\
& \quad + \tilde{K}'_T - \tilde{K}'_s - \int_s^T \tilde{Z}'_r dB_r, \quad s \in [t, T]; \\
& \text{(iii)} \tilde{Y}'_s \geq h(s, X_s^{t, \zeta}), \text{ a.s., for any } s \in [t, T]; \\
& \text{(iv)} \tilde{K}' \text{ is continuous and increasing, } \tilde{K}'_t = 0, \quad \int_t^T (\tilde{Y}'_r - h(r, X_r^{t, \zeta}))d\tilde{K}'_r = 0,
\end{aligned} \tag{6.9}$$

and

$$\begin{aligned}
& \text{(i)} \bar{Y}' \in \mathcal{S}^2(0, T; \mathbb{R}), \bar{Z}' \in \mathcal{H}^2(0, T; \mathbb{R}^d) \text{ and } \bar{K}'_T \in L^2(\Omega, \mathcal{F}_T, P); \\
& \text{(ii)} \bar{Y}'_s = \Phi(X_T^{t, \zeta}) - \mu|X_T^{t, \zeta} - X_T^{t, \zeta'}| + \mu\psi_\varepsilon(X_T^{t, \zeta} - X_T^{t, \zeta'}) + \\
& \quad \int_s^T [f(r, X_r^{t, \zeta}, \bar{Y}'_r - \mu\psi_\varepsilon(X_r^{t, \zeta} - X_r^{t, \zeta'}), \bar{Z}'_r - \mu D\psi_\varepsilon(X_r^{t, \zeta} - X_r^{t, \zeta'})(\sigma(r, X_r^{t, \zeta}) - \sigma(r, X_r^{t, \zeta'}))) \\
& \quad - \mu|X_r^{t, \zeta} - X_r^{t, \zeta'}| - \mu D\psi_\varepsilon(X_r^{t, \zeta} - X_r^{t, \zeta'})(b(r, X_r^{t, \zeta}) - b(r, X_r^{t, \zeta'})) \\
& \quad - \frac{1}{2}\mu(D^2\psi_\varepsilon(X_r^{t, \zeta} - X_r^{t, \zeta'})(\sigma(r, X_r^{t, \zeta}) - \sigma(r, X_r^{t, \zeta'})), \sigma(r, X_r^{t, \zeta}) - \sigma(r, X_r^{t, \zeta'}))]dr \\
& \quad + \bar{K}'_T - \bar{K}'_s - \int_s^T \bar{Z}'_r dB_r, \quad s \in [t, T]; \\
& \text{(iii)} \bar{Y}'_s \geq h(s, X_s^{t, \zeta}), \text{ a.s., for any } s \in [t, T]; \\
& \text{(iv)} \bar{K}' \text{ is continuous and increasing, } \bar{K}'_t = 0, \quad \int_t^T (\bar{Y}'_r - h(r, X_r^{t, \zeta}))d\bar{K}'_r = 0.
\end{aligned} \tag{6.10}$$

Obviously, also the RBSDEs (6.9) and (6.10) satisfy the assumption (H6.2) and, thus, admit unique solutions $(\tilde{Y}', \tilde{Z}', \tilde{K}')$ and $(\bar{Y}', \bar{Z}', \bar{K}')$, respectively. On the other hand, from the uniqueness of the solution of RBSDE we know that

$$\begin{aligned}
& \tilde{Y}'_s = \tilde{Y}_s - \mu\psi_\varepsilon(X_s^{t, \zeta} - X_s^{t, \zeta'}), \text{ for all } s \in [t, T], \text{ P-a.s.,} \\
& \tilde{Z}'_s = \tilde{Z}_s - \mu D\psi_\varepsilon(X_s^{t, \zeta} - X_s^{t, \zeta'})(\sigma(s, X_s^{t, \zeta}) - \sigma(s, X_s^{t, \zeta'})), \text{ dsdP-a.e. on } [t, T] \times \Omega, \\
& \tilde{K}'_s = \tilde{K}_s, \text{ for all } s \in [t, T], \text{ P-a.s.}
\end{aligned} \tag{6.11}$$

and

$$\begin{aligned}
& \bar{Y}'_s = \bar{Y}_s + \mu\psi_\varepsilon(X_s^{t, \zeta} - X_s^{t, \zeta'}), \text{ for all } s \in [t, T], \text{ P-a.s.,} \\
& \bar{Z}'_s = \bar{Z}_s + \mu D\psi_\varepsilon(X_s^{t, \zeta} - X_s^{t, \zeta'})(\sigma(s, X_s^{t, \zeta}) - \sigma(s, X_s^{t, \zeta'})), \text{ dsdP-a.e. on } [t, T] \times \Omega, \\
& \bar{K}'_s = \bar{K}_s, \text{ for all } s \in [t, T], \text{ P-a.s.}
\end{aligned} \tag{6.12}$$

Then, for the notations introduced in Lemma 2.6 we have

$$\begin{aligned}
& \Delta g(r, \tilde{Y}'_r, \tilde{Z}'_r) = f(r, X_r^{t, \zeta}, \tilde{Y}'_r + \mu\psi_\varepsilon(X_r^{t, \zeta} - X_r^{t, \zeta'}), \tilde{Z}'_r + \mu D\psi_\varepsilon(X_r^{t, \zeta} - X_r^{t, \zeta'})(\sigma(r, X_r^{t, \zeta}) - \sigma(r, X_r^{t, \zeta'}))) \\
& - f(r, X_r^{t, \zeta}, \tilde{Y}'_r - \mu\psi_\varepsilon(X_r^{t, \zeta} - X_r^{t, \zeta'}), \tilde{Z}'_r - \mu D\psi_\varepsilon(X_r^{t, \zeta} - X_r^{t, \zeta'})(\sigma(r, X_r^{t, \zeta}) - \sigma(r, X_r^{t, \zeta'}))) \\
& + 2\mu|X_r^{t, \zeta} - X_r^{t, \zeta'}| + 2\mu D\psi_\varepsilon(X_r^{t, \zeta} - X_r^{t, \zeta'})(b(r, X_r^{t, \zeta}) - b(r, X_r^{t, \zeta'})) \\
& + \mu(D^2\psi_\varepsilon(X_r^{t, \zeta} - X_r^{t, \zeta'})(\sigma(r, X_r^{t, \zeta}) - \sigma(r, X_r^{t, \zeta'})), \sigma(r, X_r^{t, \zeta}) - \sigma(r, X_r^{t, \zeta'})); \\
& \Delta \xi = \mu|X_T^{t, \zeta} - X_T^{t, \zeta'}| - \mu\psi_\varepsilon(X_T^{t, \zeta} - X_T^{t, \zeta'}); \\
& \Delta S_r = 0.
\end{aligned} \tag{6.13}$$

From (6.5) and the Lipschitz continuity of f , b and σ we get

$$\begin{aligned} |\Delta g(r, \tilde{Y}_r', \tilde{Z}_r')| &\leq C|X_r^{t,\zeta} - X_r^{t,\zeta'}| + C\varepsilon^{\frac{1}{2}}, \text{ P-a.s.}, \\ |\Delta \xi| &\leq C|X_T^{t,\zeta} - X_T^{t,\zeta'}| + C\varepsilon^{\frac{1}{2}}, \text{ P-a.s.}, \end{aligned}$$

where the constant C is independent of ε . Therefore, from Lemma 2.6 and (6.2) we get

$$E\left[\sup_{t \leq s \leq T} |\tilde{Y}_s' - \bar{Y}_s'|^2 | \mathcal{F}_t\right] \leq C|\zeta - \zeta'|^2 + C\varepsilon, \text{ P-a.s.}$$

Furthermore, from (6.8), (6.11), (6.12) and (6.2) we have

$$\begin{aligned} E[\sup_{t \leq s \leq T} |Y_s^{t,\zeta} - Y_s^{t,\zeta'}|^2 | \mathcal{F}_t] &\leq E[\sup_{t \leq s \leq T} |\tilde{Y}_s - \bar{Y}_s|^2 | \mathcal{F}_t] \\ &\leq 2E[\sup_{t \leq s \leq T} |\tilde{Y}_s' - \bar{Y}_s'|^2 | \mathcal{F}_t] + 16\mu^2(E[\sup_{t \leq s \leq T} |X_s^{t,\zeta} - X_s^{t,\zeta'}|^2 | \mathcal{F}_t] + \varepsilon) \\ &\leq C|\zeta - \zeta'|^2 + C\varepsilon, \text{ P-a.s.} \end{aligned}$$

Finally, we let ε tend to 0 to get (ii). The proof is complete. \square

Let us now introduce the random field:

$$u(t, x) = Y_s^{t,x}|_{s=t}, \quad (t, x) \in [0, T] \times \mathbb{R}^n, \quad (6.14)$$

where $Y^{t,x}$ is the solution of RBSDE (6.3) with $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$ being replaced by $x \in \mathbb{R}^n$.

As a consequence of Proposition 6.1 we have that, for all $t \in [0, T]$, P-a.s.,

$$\begin{aligned} \text{(i)} \quad &|u(t, x) - u(t, y)| \leq C|x - y|, \text{ for all } x, y \in \mathbb{R}^n; \\ \text{(ii)} \quad &|u(t, x)| \leq C(1 + |x|), \text{ for all } x \in \mathbb{R}^n. \end{aligned} \quad (6.15)$$

The random field u and $Y^{t,\zeta}$, $(t, \zeta) \in [0, T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, are related by the following theorem.

Proposition 6.2. *Under the assumptions (H6.1) and (H6.2), for any $t \in [0, T]$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we have*

$$u(t, \zeta) = Y_t^{t,\zeta}, \quad P\text{-a.s.} \quad (6.16)$$

The proof of Proposition 6.2 can be got by adapting the corresponding argument of Peng [17] to RBSDEs, we give it for the reader's convenience. It makes use of the following definition.

Definition 6.1. *For any $t \in [0, T]$, a sequence $\{A_i\}_{i=1}^N \subset \mathcal{F}_t$ (with $1 \leq N \leq \infty$) is called a partition of (Ω, \mathcal{F}_t) if $\cup_{i=1}^N A_i = \Omega$ and $A_i \cap A_j = \emptyset$, whenever $i \neq j$.*

Proof (of Proposition 6.2): We first consider the case where ζ is a simple random variable of the form

$$\zeta = \sum_{i=1}^N x_i \mathbf{1}_{A_i}, \quad (6.17)$$

where $\{A_i\}_{i=1}^N$ is a finite partition of (Ω, \mathcal{F}_t) and $x_i \in \mathbb{R}^n$, for $1 \leq i \leq N$.

For each i , we put $(X_s^i, Y_s^i, Z_s^i) \equiv (X_s^{t,x_i}, Y_s^{t,x_i}, Z_s^{t,x_i})$. Then X^i is the solution of the SDE

$$X_s^i = x_i + \int_t^s b(r, X_r^i) dr + \int_t^s \sigma(r, X_r^i) dB_r, \quad s \in [t, T],$$

and (Y^i, Z^i, K^i) is the solution of the associated RBSDE

$$\begin{aligned} Y_s^i &= \Phi(X_T^i) + \int_s^T f(r, X_r^i, Y_r^i, Z_r^i) dr + K_T^i - K_s^i - \int_s^T Z_r^i dB_r, \quad s \in [t, T], \\ Y_s^i &\geq h(s, X_s^i), \quad \int_t^T (Y_r^i - h(r, X_r^i)) dK_r^i = 0. \end{aligned}$$

The above two equations are multiplied by $\mathbf{1}_{A_i}$ and summed up with respect to i . Thus, taking into account that $\sum_i \varphi(x_i) \mathbf{1}_{A_i} = \varphi(\sum_i x_i \mathbf{1}_{A_i})$, we get

$$\sum_{i=1}^N \mathbf{1}_{A_i} X_s^i = \sum_{i=1}^N x_i \mathbf{1}_{A_i} + \int_t^s b(r, \sum_{i=1}^N \mathbf{1}_{A_i} X_r^i) dr + \int_t^s \sigma(r, \sum_{i=1}^N \mathbf{1}_{A_i} X_r^i) dB_r$$

and

$$\begin{aligned} \sum_{i=1}^N \mathbf{1}_{A_i} Y_s^i &= \Phi(\sum_{i=1}^N \mathbf{1}_{A_i} X_T^i) + \int_s^T f(r, \sum_{i=1}^N \mathbf{1}_{A_i} X_r^i, \sum_{i=1}^N \mathbf{1}_{A_i} Y_r^i, \sum_{i=1}^N \mathbf{1}_{A_i} Z_r^i) dr \\ &\quad + \sum_{i=1}^N \mathbf{1}_{A_i} K_T^i - \sum_{i=1}^N \mathbf{1}_{A_i} K_s^i - \int_s^T \sum_{i=1}^N \mathbf{1}_{A_i} Z_r^i dB_r, \\ \sum_{i=1}^N \mathbf{1}_{A_i} Y_s^i &\geq h(s, \sum_{i=1}^N \mathbf{1}_{A_i} X_s^i), \quad \int_t^T (\sum_{i=1}^N \mathbf{1}_{A_i} Y_r^i - h(r, \sum_{i=1}^N \mathbf{1}_{A_i} X_r^i)) d(\sum_{i=1}^N \mathbf{1}_{A_i} K_r^i) = 0. \end{aligned}$$

Then the strong uniqueness property of the solution of the SDE and the associated RBSDE yields

$$X_s^{t,\zeta} = \sum_{i=1}^N X_s^i \mathbf{1}_{A_i}, \quad (Y_s^{t,\zeta}, Z_s^{t,\zeta}, K_s^{t,\zeta}) = (\sum_{i=1}^N \mathbf{1}_{A_i} Y_s^i, \sum_{i=1}^N \mathbf{1}_{A_i} Z_s^i, \sum_{i=1}^N \mathbf{1}_{A_i} K_s^i), \quad s \in [t, T].$$

Finally, from $u(t, x_i) = Y_t^i$, $1 \leq i \leq N$, we deduce that

$$Y_t^{t,\zeta} = \sum_{i=1}^N Y_t^i \mathbf{1}_{A_i} = \sum_{i=1}^N u(t, x_i) \mathbf{1}_{A_i} = u(t, \sum_{i=1}^N x_i \mathbf{1}_{A_i}) = u(t, \zeta).$$

Therefore, for simple random variables, we have the desired result.

Given a general $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$ we can choose a sequence of simple random variables $\{\zeta_i\}$ which converges to ζ in $L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$. Consequently, from the estimates (6.4), (6.15) and the first step of the proof, we have

$$\begin{aligned} E|Y_t^{t,\zeta_i} - Y_t^{t,\zeta}|^2 &\leq CE|\zeta_i - \zeta|^2 \rightarrow 0, \quad i \rightarrow \infty, \\ E|u(t, \zeta_i) - u(t, \zeta)|^2 &\leq CE|\zeta_i - \zeta|^2 \rightarrow 0, \quad i \rightarrow \infty, \\ \text{and} \quad Y_t^{t,\zeta_i} &= u(t, \zeta_i), \quad i \geq 1. \end{aligned}$$

Then the proof is complete. \square

7 Appendix II: Complement to Section 3

We begin with the

Proof of Proposition 3.1. We recall that $\Omega = C_0([0, T]; \mathbb{R}^d)$ and denote by H the Cameron-Martin space of all absolutely continuous elements $h \in \Omega$ whose derivative \dot{h} belongs to $L^2([0, T], \mathbb{R}^d)$. For any $h \in H$, we define the mapping $\tau_h \omega := \omega + h$, $\omega \in \Omega$. Obviously, $\tau_h : \Omega \rightarrow \Omega$ is a bijection

and its law is given by $P \circ [\tau_h]^{-1} = \exp\{\int_0^T \dot{h}_s dB_s - \frac{1}{2} \int_0^T |\dot{h}_s|^2 ds\}P$. Let $(t, x) \in [0, T] \times \mathbb{R}^n$ be arbitrarily fixed and put $H_t = \{h \in H | h(\cdot) = h(\cdot \wedge t)\}$. We split now the proof in the following steps:

1st step: For any $u \in \mathcal{U}_{t,T}$, $v \in \mathcal{V}_{t,T}$, $h \in H_t$, $J(t, x; u, v)(\tau_h) = J(t, x; u(\tau_h), v(\tau_h))$, P-a.s.

Indeed, for $h \in H_t$ we apply the Girsanov transformation to SDE (3.1) (with $\zeta = x$). Notice that since $h \in H_t$, we have $dB_s(\tau_h) = dB_s$, $s \in [t, T]$. We compare the thus obtained equation with the SDE got from (3.1) by substituting the transformed control processes $u(\tau_h), v(\tau_h)$ for u and v . Then, from the uniqueness of the solution of (3.1) we get $X_s^{t,x;u,v}(\tau_h) = X_s^{t,x;u(\tau_h),v(\tau_h)}$, for any $s \in [t, T]$, P-a.s. Furthermore, by a similar Girsanov transformation argument we get from the uniqueness of the solution of RBSDE (3.5),

$$Y_s^{t,x;u,v}(\tau_h) = Y_s^{t,x;u(\tau_h),v(\tau_h)}, \text{ for any } s \in [t, T], \text{ P-a.s.},$$

$$Z_s^{t,x;u,v}(\tau_h) = Z_s^{t,x;u(\tau_h),v(\tau_h)}, \text{ dsdP-a.e. on } [t, T] \times \Omega,$$

$$K_s^{t,x;u,v}(\tau_h) = K_s^{t,x;u(\tau_h),v(\tau_h)}, \text{ for any } s \in [t, T], \text{ P-a.s.}$$

This implies, in particular, that

$$J(t, x; u, v)(\tau_h) = J(t, x; u(\tau_h), v(\tau_h)), \text{ P-a.s.}$$

2nd step: For $\beta \in \mathcal{B}_{t,T}$, $h \in H_t$, let $\beta^h(u) := \beta(u(\tau_{-h}))(\tau_h)$, $u \in \mathcal{U}_{t,T}$. Then $\beta^h \in \mathcal{B}_{t,T}$.

Obviously, β^h maps $\mathcal{U}_{t,T}$ into $\mathcal{V}_{t,T}$. Moreover, this mapping is nonanticipating. Indeed, let $S : \Omega \rightarrow [t, T]$ be an \mathcal{F}_r -stopping time and $u_1, u_2 \in \mathcal{U}_{t,T}$ with $u_1 \equiv u_2$ on $\llbracket t, S \rrbracket$. Then, obviously, $u_1(\tau_{-h}) \equiv u_2(\tau_{-h})$ on $\llbracket t, S(\tau_{-h}) \rrbracket$ (notice that $S(\tau_{-h})$ is still a stopping time), and because $\beta \in \mathcal{B}_{t,T}$ we have $\beta(u_1(\tau_{-h})) \equiv \beta(u_2(\tau_{-h}))$ on $\llbracket t, S(\tau_{-h}) \rrbracket$. Therefore,

$$\beta^h(u_1) = \beta(u_1(\tau_{-h}))(\tau_h) \equiv \beta(u_2(\tau_{-h}))(\tau_h) = \beta^h(u_2) \text{ on } \llbracket t, S \rrbracket.$$

3rd step: For all $h \in H_t$ and $\beta \in \mathcal{B}_{t,T}$ we have:

$$\{\text{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u, \beta(u))\}(\tau_h) = \text{esssup}_{u \in \mathcal{U}_{t,T}} \{J(t, x; u, \beta(u))(\tau_h)\}, \text{ P-a.s.}$$

Indeed, with the notation $I(t, x, \beta) := \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u, \beta(u))$, $\beta \in \mathcal{B}_{t,T}$, we have $I(t, x, \beta) \geq J(t, x; u, \beta(u))$, and thus $I(t, x, \beta)(\tau_h) \geq J(t, x; u, \beta(u))(\tau_h)$, P-a.s., for all $u \in \mathcal{U}_{t,T}$. On the other hand, for any random variable ζ satisfying $\zeta \geq J(t, x; u, \beta(u))(\tau_h)$ and hence also $\zeta(\tau_{-h}) \geq J(t, x; u, \beta(u))$, P-a.s., for all $u \in \mathcal{U}_{t,T}$, we have $\zeta(\tau_{-h}) \geq I(t, x, \beta)$, P-a.s., i.e., $\zeta \geq I(t, x, \beta)(\tau_h)$, P-a.s. Consequently,

$$I(t, x, \beta)(\tau_h) = \text{esssup}_{u \in \mathcal{U}_{t,T}} \{J(t, x; u, \beta(u))(\tau_h)\}, \text{ P-a.s.}$$

4th step: $W(t, x)$ is invariant with respect to the Girsanov transformation τ_h , i.e.,

$$W(t, x)(\tau_h) = W(t, x), \text{ P-a.s., for any } h \in H.$$

Let us first assume that $h \in H_t$. Then, similarly to the third step we can show that for all $h \in H_t$,

$$\{\text{essinf}_{\beta \in \mathcal{B}_{t,T}} I(t, x; \beta)\}(\tau_h) = \text{essinf}_{\beta \in \mathcal{B}_{t,T}} \{I(t, x; \beta)(\tau_h)\}, \text{ P-a.s.}$$

Then, using the results of the former three steps we have, for any $h \in H_t$,

$$\begin{aligned}
W(t, x)(\tau_h) &= \operatorname{essinf}_{\beta \in \mathcal{B}_{t,T}} \operatorname{esssup}_{u \in \mathcal{U}_{t,T}} \{J(t, x; u, \beta(u))(\tau_h)\} \\
&= \operatorname{essinf}_{\beta \in \mathcal{B}_{t,T}} \operatorname{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u(\tau_h), \beta^h(u(\tau_h))) \\
&= \operatorname{essinf}_{\beta \in \mathcal{B}_{t,T}} \operatorname{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u, \beta^h(u)) \\
&= \operatorname{essinf}_{\beta \in \mathcal{B}_{t,T}} \operatorname{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u, \beta(u)) \\
&= W(t, x), \text{ P-a.s.},
\end{aligned}$$

where we have used the relations $\{u(\tau_h)|u(\cdot) \in \mathcal{U}_{t,T}\} = \mathcal{U}_{t,T}$, $\{\beta^h|\beta \in \mathcal{B}_{t,T}\} = \mathcal{B}_{t,T}$ in order to obtain the both latter equalities. Therefore, for any $h \in H_t$, $W(t, x)(\tau_h) = W(t, x)$, P-a.s., and since $W(t, x)$ is \mathcal{F}_t -measurable, we have this relation even for all $h \in H$. Indeed, recall that our underlying fundamental space is $\Omega = C_0([0, T]; \mathbb{R}^d)$ and that, due to the definition of the filtration, the \mathcal{F}_t -measurable random variable $W(t, x)(\omega)$, $\omega \in \Omega$, depends only on the restriction of ω to the time interval $[0, t]$.

The result of the 4th step combined with the following auxiliary Lemma 7.1 completes our proof. \square

Lemma 7.1. *Let ζ be a random variable defined over our classical Wiener space $(\Omega, \mathcal{F}_T, P)$, such that $\zeta(\tau_h) = \zeta$, P-a.s., for any $h \in H$. Then $\zeta = E\zeta$, P-a.s.*

The proof of Lemma 7.1 can be found in Buckdahn and Li [3].

Let us come now to the

Proof of Theorem 3.1. To simplify notations we put

$$W_\delta(t, x) = \operatorname{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \operatorname{esssup}_{u \in \mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{t,x;u,\beta(u)}[W(t+\delta, X_{t+\delta}^{t,x;u,\beta(u)})].$$

The proof that $W_\delta(t, x)$ coincides with $W(t, x)$ will be split into a sequel of lemmas which all suppose that (H3.1) and (H3.2) are satisfied. Let us fix $(t, x) \in [0, T] \times \mathbb{R}^n$.

Lemma 7.2. *$W_\delta(t, x)$ is deterministic.*

The proof of this lemma uses the same ideas as that of Proposition 3.1 so that it can be omitted here. \square

Lemma 7.3. $W_\delta(t, x) \leq W(t, x)$.

Proof. Let $\beta \in \mathcal{B}_{t,T}$ be arbitrarily fixed. Then, given a $u_2(\cdot) \in \mathcal{U}_{t+\delta,T}$, we define as follows the restriction β_1 of β to $\mathcal{U}_{t,t+\delta}$:

$$\beta_1(u_1) := \beta(u_1 \oplus u_2)|_{[t,t+\delta]}, \quad u_1(\cdot) \in \mathcal{U}_{t,t+\delta},$$

where $u_1 \oplus u_2 := u_1 \mathbf{1}_{[t,t+\delta]} + u_2 \mathbf{1}_{(t+\delta,T]}$ extends $u_1(\cdot)$ to an element of $\mathcal{U}_{t,T}$. It is easy to check that $\beta_1 \in \mathcal{B}_{t,t+\delta}$. Moreover, from the nonanticipativity property of β we deduce that β_1 is independent of the special choice of $u_2(\cdot) \in \mathcal{U}_{t+\delta,T}$. Consequently, from the definition of $W_\delta(t, x)$,

$$W_\delta(t, x) \leq \operatorname{esssup}_{u_1 \in \mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{t,x;u_1,\beta_1(u_1)}[W(t+\delta, X_{t+\delta}^{t,x;u_1,\beta_1(u_1)})], \text{ P-a.s.} \quad (7.1)$$

We use the notation $I_\delta(t, x, u, v) := G_{t, t+\delta}^{t, x; u, v}[W(t+\delta, X_{t+\delta}^{t, x; u, v})]$ and notice that there exists a sequence $\{u_i^1, i \geq 1\} \subset \mathcal{U}_{t, t+\delta}$ such that

$$I_\delta(t, x, \beta_1) := \text{esssup}_{u_1 \in \mathcal{U}_{t, t+\delta}} I_\delta(t, x, u_1, \beta_1(u_1)) = \sup_{i \geq 1} I_\delta(t, x, u_i^1, \beta_1(u_i^1)), \quad \text{P-a.s.}$$

For any $\varepsilon > 0$, we put $\tilde{\Gamma}_i := \{I_\delta(t, x, \beta_1) \leq I_\delta(t, x, u_i^1, \beta_1(u_i^1)) + \varepsilon\} \in \mathcal{F}_t, i \geq 1$. Then $\Gamma_1 := \tilde{\Gamma}_1, \Gamma_i := \tilde{\Gamma}_i \setminus (\cup_{l=1}^{i-1} \tilde{\Gamma}_l) \in \mathcal{F}_t, i \geq 2$, form an (Ω, \mathcal{F}_t) -partition, and $u_1^\varepsilon := \sum_{i \geq 1} \mathbf{1}_{\Gamma_i} u_i^1$ belongs obviously to $\mathcal{U}_{t, t+\delta}$. Moreover, from the nonanticipativity of β_1 we have $\beta_1(u_1^\varepsilon) = \sum_{i \geq 1} \mathbf{1}_{\Gamma_i} \beta_1(u_i^1)$, and from the uniqueness of the solution of SDE (3.1) and RBSDE (3.5), we deduce that $I_\delta(t, x, u_1^\varepsilon, \beta_1(u_1^\varepsilon)) = \sum_{i \geq 1} \mathbf{1}_{\Gamma_i} I_\delta(t, x, u_i^1, \beta_1(u_i^1))$, P-a.s. Hence,

$$\begin{aligned} W_\delta(t, x) \leq I_\delta(t, x, \beta_1) &\leq \sum_{i \geq 1} \mathbf{1}_{\Gamma_i} I_\delta(t, x, u_i^1, \beta_1(u_i^1)) + \varepsilon = I_\delta(t, x, u_1^\varepsilon, \beta_1(u_1^\varepsilon)) + \varepsilon \\ &= G_{t, t+\delta}^{t, x; u_1^\varepsilon, \beta_1(u_1^\varepsilon)}[W(t+\delta, X_{t+\delta}^{t, x; u_1^\varepsilon, \beta_1(u_1^\varepsilon)})] + \varepsilon, \quad \text{P-a.s.} \end{aligned} \quad (7.2)$$

On the other hand, using the fact that $\beta_1(\cdot) := \beta(\cdot \oplus u_2) \in \mathcal{B}_{t, t+\delta}$ does not depend on $u_2(\cdot) \in \mathcal{U}_{t+\delta, T}$ we can define $\beta_2(u_2) := \beta(u_1^\varepsilon \oplus u_2)|_{[t+\delta, T]}$, for all $u_2(\cdot) \in \mathcal{U}_{t+\delta, T}$. The such defined $\beta_2 : \mathcal{U}_{t+\delta, T} \rightarrow \mathcal{V}_{t+\delta, T}$ belongs to $\mathcal{B}_{t+\delta, T}$ since $\beta \in \mathcal{B}_{t, T}$. Therefore, from the definition of $W(t+\delta, y)$ we have, for any $y \in \mathbb{R}^n$,

$$W(t+\delta, y) \leq \text{esssup}_{u_2 \in \mathcal{U}_{t+\delta, T}} J(t+\delta, y; u_2, \beta_2(u_2)), \quad \text{P-a.s.}$$

Finally, because there exists a constant $C \in \mathbb{R}$ such that

$$\begin{aligned} \text{(i)} \quad & |W(t+\delta, y) - W(t+\delta, y')| \leq C|y - y'|, \text{ for any } y, y' \in \mathbb{R}^n; \\ \text{(ii)} \quad & |J(t+\delta, y, u_2, \beta_2(u_2)) - J(t+\delta, y', u_2, \beta_2(u_2))| \leq C|y - y'|, \text{ P-a.s.,} \\ & \text{for any } u_2 \in \mathcal{U}_{t+\delta, T}, \end{aligned} \quad (7.3)$$

(see Lemma 3.2-(i) and (3.6)-(i)) we can show by approximating $X_{t+\delta}^{t, x; u_1^\varepsilon, \beta_1(u_1^\varepsilon)}$ that

$$W(t+\delta, X_{t+\delta}^{t, x; u_1^\varepsilon, \beta_1(u_1^\varepsilon)}) \leq \text{esssup}_{u_2 \in \mathcal{U}_{t+\delta, T}} J(t+\delta, X_{t+\delta}^{t, x; u_1^\varepsilon, \beta_1(u_1^\varepsilon)}; u_2, \beta_2(u_2)), \quad \text{P-a.s.}$$

To estimate the right side of the latter inequality we note that there exists some sequence $\{u_j^2, j \geq 1\} \subset \mathcal{U}_{t+\delta, T}$ such that

$$\text{esssup}_{u_2 \in \mathcal{U}_{t+\delta, T}} J(t+\delta, X_{t+\delta}^{t, x; u_1^\varepsilon, \beta_1(u_1^\varepsilon)}; u_2, \beta_2(u_2)) = \sup_{j \geq 1} J(t+\delta, X_{t+\delta}^{t, x; u_1^\varepsilon, \beta_1(u_1^\varepsilon)}; u_j^2, \beta_2(u_j^2)), \quad \text{P-a.s.}$$

Then, putting

$\tilde{\Delta}_j := \{\text{esssup}_{u_2 \in \mathcal{U}_{t+\delta, T}} J(t+\delta, X_{t+\delta}^{t, x; u_1^\varepsilon, \beta_1(u_1^\varepsilon)}; u_2, \beta_2(u_2)) \leq J(t+\delta, X_{t+\delta}^{t, x; u_1^\varepsilon, \beta_1(u_1^\varepsilon)}; u_j^2, \beta_2(u_j^2)) + \varepsilon\} \in \mathcal{F}_{t+\delta}, j \geq 1$; we have with $\Delta_1 := \tilde{\Delta}_1, \Delta_j := \tilde{\Delta}_j \setminus (\cup_{l=1}^{j-1} \tilde{\Delta}_l) \in \mathcal{F}_{t+\delta}, j \geq 2$, an $(\Omega, \mathcal{F}_{t+\delta})$ -partition and $u_2^\varepsilon := \sum_{j \geq 1} \mathbf{1}_{\Delta_j} u_j^2 \in \mathcal{U}_{t+\delta, T}$. From the nonanticipativity of β_2 we have $\beta_2(u_2^\varepsilon) = \sum_{j \geq 1} \mathbf{1}_{\Delta_j} \beta_2(u_j^2)$ and from the definition of β_1, β_2 we know that $\beta(u_1^\varepsilon \oplus u_2^\varepsilon) = \beta_1(u_1^\varepsilon) \oplus \beta_2(u_2^\varepsilon)$. Thus, again from the uniqueness of the solution of our FBSDE, we get

$$\begin{aligned} J(t+\delta, X_{t+\delta}^{t, x; u_1^\varepsilon, \beta_1(u_1^\varepsilon)}; u_2^\varepsilon, \beta_2(u_2^\varepsilon)) &= Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t, x; u_1^\varepsilon, \beta_1(u_1^\varepsilon)}; u_2^\varepsilon, \beta_2(u_2^\varepsilon)} \quad (\text{see (3.8)}) \\ &= \sum_{j \geq 1} \mathbf{1}_{\Delta_j} Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t, x; u_1^\varepsilon, \beta_1(u_1^\varepsilon)}; u_j^2, \beta_2(u_j^2)} \\ &= \sum_{j \geq 1} \mathbf{1}_{\Delta_j} J(t+\delta, X_{t+\delta}^{t, x; u_1^\varepsilon, \beta_1(u_1^\varepsilon)}; u_j^2, \beta_2(u_j^2)), \quad \text{P-a.s.} \end{aligned}$$

Consequently,

$$\begin{aligned}
W(t + \delta, X_{t+\delta}^{t,x;u_1^\varepsilon, \beta_1(u_1^\varepsilon)}) &\leq \text{esssup}_{u_2 \in \mathcal{U}_{t+\delta, T}} J(t + \delta, X_{t+\delta}^{t,x;u_1^\varepsilon, \beta_1(u_1^\varepsilon)}; u_2, \beta_2(u_2)) \\
&\leq \sum_{j \geq 1} \mathbf{1}_{\Delta_j} Y_{t+\delta}^{t,x;u_1^\varepsilon \oplus u_j^2, \beta(u_1^\varepsilon \oplus u_j^2)} + \varepsilon \\
&= Y_{t+\delta}^{t,x;u_1^\varepsilon \oplus u_2^\varepsilon, \beta(u_1^\varepsilon \oplus u_2^\varepsilon)} + \varepsilon \\
&= Y_{t+\delta}^{t,x;u^\varepsilon, \beta(u^\varepsilon)} + \varepsilon, \text{ P-a.s.},
\end{aligned} \tag{7.4}$$

where $u^\varepsilon := u_1^\varepsilon \oplus u_2^\varepsilon \in \mathcal{U}_{t,T}$. From (7.2), (7.4), Lemma 2.4 (comparison theorem for RBSDEs) and Lemma 2.6 we have

$$\begin{aligned}
W_\delta(t, x) &\leq G_{t,t+\delta}^{t,x;u_1^\varepsilon, \beta_1(u_1^\varepsilon)} [Y_{t+\delta}^{t,x;u^\varepsilon, \beta(u^\varepsilon)} + \varepsilon] + \varepsilon \\
&\leq G_{t,t+\delta}^{t,x;u_1^\varepsilon, \beta_1(u_1^\varepsilon)} [Y_{t+\delta}^{t,x;u^\varepsilon, \beta(u^\varepsilon)}] + (C + 1)\varepsilon \\
&= G_{t,t+\delta}^{t,x;u^\varepsilon, \beta(u^\varepsilon)} [Y_{t+\delta}^{t,x;u^\varepsilon, \beta(u^\varepsilon)}] + (C + 1)\varepsilon \\
&= Y_t^{t,x;u^\varepsilon, \beta(u^\varepsilon)} + (C + 1)\varepsilon \\
&\leq \text{esssup}_{u \in \mathcal{U}_{t,T}} Y_t^{t,x;u, \beta(u)} + (C + 1)\varepsilon, \text{ P-a.s.}
\end{aligned} \tag{7.5}$$

Since $\beta \in \mathcal{B}_{t,T}$ has been arbitrarily chosen we have (7.5) for all $\beta \in \mathcal{B}_{t,T}$. Therefore,

$$W_\delta(t, x) \leq \text{essinf}_{\beta \in \mathcal{B}_{t,T}} \text{esssup}_{u \in \mathcal{U}_{t,T}} Y_t^{t,x;u, \beta(u)} + (C + 1)\varepsilon = W(t, x) + (C + 1)\varepsilon. \tag{7.6}$$

Finally, letting $\varepsilon \downarrow 0$, we get $W_\delta(t, x) \leq W(t, x)$. \square

Lemma 7.4. $W(t, x) \leq W_\delta(t, x)$.

Proof. We continue to use the notations introduced above. From the definition of $W_\delta(t, x)$ we have

$$\begin{aligned}
W_\delta(t, x) &= \text{essinf}_{\beta_1 \in \mathcal{B}_{t,t+\delta}} \text{esssup}_{u_1 \in \mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{t,x;u_1, \beta_1(u_1)} [W(t + \delta, X_{t+\delta}^{t,x;u_1, \beta_1(u_1)})] \\
&= \text{essinf}_{\beta_1 \in \mathcal{B}_{t,t+\delta}} I_\delta(t, x, \beta_1),
\end{aligned}$$

and, for some sequence $\{\beta_i^1, i \geq 1\} \subset \mathcal{B}_{t,t+\delta}$,

$$W_\delta(t, x) = \inf_{i \geq 1} I_\delta(t, x, \beta_i^1), \text{ P-a.s.}$$

For any $\varepsilon > 0$, we let $\tilde{\Lambda}_i := \{I_\delta(t, x, \beta_i^1) - \varepsilon \leq W_\delta(t, x)\} \in \mathcal{F}_t$, $i \geq 1$, $\Lambda_1 := \tilde{\Lambda}_1$ and $\Lambda_i := \tilde{\Lambda}_i \setminus (\cup_{l=1}^{i-1} \tilde{\Lambda}_l) \in \mathcal{F}_t$, $i \geq 2$. Then $\{\Lambda_i, i \geq 1\}$ is an (Ω, \mathcal{F}_t) -partition, $\beta_1^\varepsilon := \sum_{i \geq 1} \mathbf{1}_{\Lambda_i} \beta_i^1$ belongs to $\mathcal{B}_{t,t+\delta}$, and from the uniqueness of the solution of our FBSDE we conclude that $I_\delta(t, x, u_1, \beta_1^\varepsilon(u_1)) = \sum_{i \geq 1} \mathbf{1}_{\Lambda_i} I_\delta(t, x, u_1, \beta_i^1(u_1))$, P-a.s., for all $u_1(\cdot) \in \mathcal{U}_{t,t+\delta}$. Hence,

$$\begin{aligned}
W_\delta(t, x) &\geq \sum_{i \geq 1} \mathbf{1}_{\Lambda_i} I_\delta(t, x, \beta_i^1) - \varepsilon \\
&\geq \sum_{i \geq 1} \mathbf{1}_{\Lambda_i} I_\delta(t, x, u_1, \beta_i^1(u_1)) - \varepsilon \\
&= I_\delta(t, x, u_1, \beta_1^\varepsilon(u_1)) - \varepsilon \\
&= G_{t,t+\delta}^{t,x;u_1, \beta_1^\varepsilon(u_1)} [W(t + \delta, X_{t+\delta}^{t,x;u_1, \beta_1^\varepsilon(u_1)})] - \varepsilon, \text{ P-a.s., for all } u_1 \in \mathcal{U}_{t,t+\delta}.
\end{aligned} \tag{7.7}$$

On the other hand, from the definition of $W(t + \delta, y)$, with the same technique as before, we deduce that, for any $y \in \mathbb{R}^n$, there exists $\beta_y^\varepsilon \in \mathcal{B}_{t+\delta, T}$ such that

$$W(t + \delta, y) \geq \text{esssup}_{u_2 \in \mathcal{U}_{t+\delta, T}} J(t + \delta, y; u_2, \beta_y^\varepsilon(u_2)) - \varepsilon, \text{ P-a.s.} \tag{7.8}$$

Let $\{O_i\}_{i \geq 1} \subset \mathcal{B}(\mathbb{R}^n)$ be a decomposition of \mathbb{R}^n such that $\sum_{i \geq 1} O_i = \mathbb{R}^n$ and $\text{diam}(O_i) \leq \varepsilon$, $i \geq 1$. And let y_i be an arbitrarily fixed element of O_i , $i \geq 1$. Defining $[X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}] := \sum_{i \geq 1} y_i \mathbf{1}_{\{X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)} \in O_i\}}$, we have

$$|X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)} - [X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}]| \leq \varepsilon, \text{ everywhere on } \Omega, \text{ for all } u_1 \in \mathcal{U}_{t,t+\delta}. \quad (7.9)$$

Moreover, for each y_i , there exists some $\beta_{y_i}^\varepsilon \in \mathcal{B}_{t+\delta,T}$ such that (7.8) holds, and, clearly, $\beta_{u_1}^\varepsilon := \sum_{i \geq 1} \mathbf{1}_{\{X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)} \in O_i\}} \beta_{y_i}^\varepsilon \in \mathcal{B}_{t+\delta,T}$.

Now we can define the new strategy $\beta^\varepsilon(u) := \beta_1^\varepsilon(u_1) \oplus \beta_{u_1}^\varepsilon(u_2)$, $u \in \mathcal{U}_{t,T}$, where $u_1 = u|_{[t,t+\delta]}$, $u_2 = u|_{(t+\delta,T]}$ (restriction of u to $[t,t+\delta] \times \Omega$ and $(t+\delta,T] \times \Omega$, resp.). Obviously, β^ε maps $\mathcal{U}_{t,T}$ into $\mathcal{V}_{t,T}$. Moreover, β^ε is nonanticipating: Indeed, let $S : \Omega \rightarrow [t,T]$ be an \mathcal{F}_r -stopping time and $u, u' \in \mathcal{U}_{t,T}$ be such that $u \equiv u'$ on $\llbracket t, S \rrbracket$. Decomposing u, u' into $u_1, u'_1 \in \mathcal{U}_{t,t+\delta}$, $u_2, u'_2 \in \mathcal{U}_{t+\delta,T}$ such that $u = u_1 \oplus u_2$ and $u' = u'_1 \oplus u'_2$ we have $u_1 \equiv u'_1$ on $\llbracket t, S \wedge (t+\delta) \rrbracket$, from where we get $\beta_1^\varepsilon(u_1) \equiv \beta_1^\varepsilon(u'_1)$ on $\llbracket t, S \wedge (t+\delta) \rrbracket$ (recall that β_1^ε is nonanticipating). On the other hand, $u_2 \equiv u'_2$ on $\llbracket t+\delta, S \vee (t+\delta) \rrbracket \subset (t+\delta, T] \times \{S > t+\delta\}$, and on $\{S > t+\delta\}$ we have $X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)} = X_{t+\delta}^{t,x;u'_1,\beta_1^\varepsilon(u'_1)}$. Consequently, from our definition, $\beta_{u_1}^\varepsilon = \beta_{u'_1}^\varepsilon$ on $\{S > t+\delta\}$ and $\beta_{u_1}^\varepsilon(u_2) \equiv \beta_{u'_1}^\varepsilon(u'_2)$ on $\llbracket t+\delta, S \vee (t+\delta) \rrbracket$. This yields $\beta^\varepsilon(u) = \beta_1^\varepsilon(u_1) \oplus \beta_{u_1}^\varepsilon(u_2) \equiv \beta_1^\varepsilon(u'_1) \oplus \beta_{u'_1}^\varepsilon(u'_2) = \beta^\varepsilon(u')$ on $\llbracket t, S \rrbracket$, from where it follows that $\beta^\varepsilon \in \mathcal{B}_{t,T}$.

Let now $u \in \mathcal{U}_{t,T}$ be arbitrarily chosen and decomposed into $u_1 = u|_{[t,t+\delta]} \in \mathcal{U}_{t,t+\delta}$ and $u_2 = u|_{(t+\delta,T]} \in \mathcal{U}_{t+\delta,T}$. Then, from (7.7), (7.3)-(i), (7.9) and Lemma 2.6 we obtain,

$$\begin{aligned} W_\delta(t, x) &\geq G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[W(t+\delta, X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)})] - \varepsilon \\ &\geq G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[W(t+\delta, [X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}]) + C\varepsilon] - C'\varepsilon \\ &= G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[\sum_{i \geq 1} \mathbf{1}_{\{X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)} \in O_i\}} W(t+\delta, y_i) + C\varepsilon] - C'\varepsilon, \text{ P-a.s.} \end{aligned} \quad (7.10)$$

Furthermore, from (7.8), (7.3)-(ii), (7.9), Lemmata 2.4 and 2.6, we have

$$\begin{aligned} W_\delta(t, x) &\geq G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[\sum_{i \geq 1} \mathbf{1}_{\{X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)} \in O_i\}} J(t+\delta, y_i; u_2, \beta_{y_i}^\varepsilon(u_2)) + C\varepsilon] - C'\varepsilon \\ &= G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[J(t+\delta, [X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}]; u_2, \beta_{u_1}^\varepsilon(u_2)) + C\varepsilon] - C'\varepsilon \\ &\geq G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[J(t+\delta, X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}; u_2, \beta_{u_1}^\varepsilon(u_2))] - C\varepsilon \\ &= G_{t,t+\delta}^{t,x;u,\beta^\varepsilon(u)}[Y_{t+\delta}^{t,x,u,\beta^\varepsilon(u)}] - C\varepsilon \\ &= Y_t^{t,x,u,\beta^\varepsilon(u)} - C\varepsilon, \text{ P-a.s., for any } u \in \mathcal{U}_{t,T}. \end{aligned} \quad (7.11)$$

Here, the constants C and C' vary from line to line. Consequently,

$$\begin{aligned} W_\delta(t, x) &\geq \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u, \beta^\varepsilon(u)) - C\varepsilon \\ &\geq \text{essinf}_{\beta \in \mathcal{B}_{t,T}} \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u, \beta(u)) - C\varepsilon \\ &= W(t, x) - C\varepsilon, \text{ P-a.s.} \end{aligned} \quad (7.12)$$

Finally, letting $\varepsilon \downarrow 0$ we get $W_\delta(t, x) \geq W(t, x)$. The proof is complete. \square

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